THE FLUCTUATION BEHAVIOUR OF THE STOCHASTIC POINT VORTEX MODEL WITH COMMON NOISE

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ABSTRACT. This article studies the fluctuation behaviour of the stochastic point vortex model with common noise. Using the martingale method combined with a localization argument, we prove that the sequence of fluctuation processes converges in distribution to the unique probabilistically strong solution of a linear stochastic evolution equation. In particular, we establish the strong convergence from the stochastic point vortex model with common noise to the conditional McKean-Vlasov equation.

CONTENTS

1. INTRODUCTION

In this article, we investigate the fluctuation behavior of the following weakly interacting particle system with common noise on torus $\mathbb{T}^2 = [-\pi, \pi]^2$,

$$X_i^N(t) = X_i(0) + \frac{1}{N}\sum_{j\neq i}\int_0^t K(X_i^N(s) - X_j^N(s))\mathrm{d}s + \sqrt{2}B_i(t) + \int_0^t \sigma(X_i^N(s)) \circ \mathrm{d}W_s, \quad t \in [0, T]. \quad (1.1)$$

This system is commonly referred to as the stochastic point vortex model. Here $T > 0, N \in \mathbb{N}$ and $\circ$ denotes the stochastic integral in the Stratonovich sense. The interaction between particles is defined

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by the Biot-Savart kernel K on $\mathbb{T}^2$, namely

$$K(x) = -\frac{1}{2\pi}\frac{x^\perp}{|x|^2} + K_0(x), \quad x^\perp = (x_2, -x_1), \quad x = (x_1, x_2) \in \mathbb{T}^2, \tag{1.2}$$

where K_0 is a smooth correction to periodize K on torus $\mathbb{T}^2 = [-\pi, \pi]^2$. The common noise shared by all particles is described by the term $\int_0^\cdot \sigma(X_i(s)) \circ \mathrm{d}W_s$, where σ is a smooth and divergence free vector field and $\{W_t, t \in [0,T]\}$ represents a 1-dimensional standard Brownian motion. Additionally, $\{B_i, i \in \mathbb{N}\}$ are independent 2-dimensional Brownian motions on torus $\mathbb{T}^2$, modeling the individual noise for each particle. The initial positions of the particles are given by a sequence of independent and identically distributed (i.i.d.) random variables $\{X_i^N(0), i \in \mathbb{N}\}$ taking values in $\mathbb{T}^2$. The identical distribution for initial values $\{X_i^N(0), i \in \mathbb{N}\}$ is denoted by $\mathcal{L}(X(0))$. We further assume that the initial positions $\{X_i^N(0), i \in \mathbb{N}\}$, the individual noises $\{B_i, i \in \mathbb{N}\}$ and environmental noise W are mutually independent.

The aim of this paper is to study the asymptotic behavior of the fluctuation process

$$\eta_t^N := \sqrt{N}(\mu_N(t) - v_t) = \frac{1}{\sqrt{N}}\sum_{i=1}^N \left(\delta_{X_i^N(t)} - v_t\right), \quad \forall t \in [0,T], \tag{1.3}$$

which describes the deviations of the empirical measures of the stochastic point vortex model (1.1)

$$\mu_N(t) := \frac{\sum_{i=1}^N \delta_{X_i^N(t)}}{N}, \quad \forall t \in [0,T] \tag{1.4}$$

from the mean field limit $(v_t)_{t\in[0,T]}$. Here $(v_t)_{t\in[0,T]}$ is the unique probabilistically strong solution to the following stochastic 2-dimensional Navier-Stokes equation (1.5) on $[0,T] \times \mathbb{T}^2$ in the sense of Definitions 2.2 and 2.3 below :

$$\mathrm{d}v = \Big(\Delta v - K * v \cdot \nabla v\Big)\mathrm{d}t - \sigma \cdot \nabla v \circ \mathrm{d}W_t, \quad v(0,\cdot) = v_0. \tag{1.5}$$

This study is often referred to as the central limit theorem for interacting particle systems (1.1). Compared to the mean field limit result (1.6), it provides a more precise description of the relationship between the interacting particle system (1.1) and the mean field limit equation (1.5), i.e, formally

$$\mu_N \overset{\mathrm{d}}{\approx} v + \frac{1}{\sqrt{N}}\eta,$$

where $\overset{\mathrm{d}}{\approx}$ means that the approximation holds in distribution and η represents the limiting process of the sequence $\{\eta^N\}_{N\in\mathbb{N}}$ in the sense of convergence in distribution. Furthermore, η is typically Gaussian-distributed in the absence of environmental noise W.

The classical mean field limit for particle systems without common noise, such as (1.1) with $\sigma = 0$, has been extensively studied over the past decade. The goal of studying the mean field limit is to analyze the asymptotic independence of particles. This result can be expressed in two forms, which are qualitatively equivalent, as established in [Szn91]. One form is that, for every $t \in [0,T]$, the empirical measure of the particle system $\mu_N(t)$ satisfies the weak convergence of measures,

$$\mu_N(t) \rightharpoonup v_t. \tag{1.6}$$

The other form, often referred to as *propagation of chaos*, states that, for fixed $k \in \mathbb{N}, t \in [0,T]$,

$$\mathcal{L}^{N,k}(t) \rightharpoonup v_t^{\otimes k},$$

where $\mathcal{L}^{N,k}(t)$ is the k-marginal of the particle distribution $\mathcal{L}(X_1^N(t), \cdots, X_N^N(t))$ for (1.1) with $\sigma = 0$. The limiting measure $(v_t)_{t\in[0,T]}$ is the solution to the following nonlinear Fokker-Planck equation

$$\partial_t v = \Delta v - K * v \cdot \nabla v,$$

and coincides with the law of a solution to the related Mckean Vlasov equation. We refer readers to the works [Osa86, Szn91, FHM14, Due16, JW18, Ser20, Lac23, BJW23, FW23, GLBM24, Wan24, CFG+24] for classical results on mean field limit and reference therein. For interacting particle systems

(1.1) with various kernels K, particular attention has been paid to systems with singular kernels due to their physical relevance. One of the most well-known examples is *the Biot-Savart kernel* K, which is also the focus of our study in this paper. The corresponding particle system is often referred to as the stochastic point vortex model, which describes the behavior of fluid. Recent advancements in this area have highlighted the relative entropy method, which not only ensures convergence results for particle systems (1.1) with singular kernels but also provides quantitative convergence rates. In our study, we address the central limit problem for the stochastic point vortex model, leveraging mean field limit results expressed through the relative entropy framework. Additionally, we use this method to derive uniform estimates. The global relative entropy method was developed by Jabin and Wang in [JW16] for second-order systems with bounded kernels and in [JW18] for first-order systems with general $W^{-1,\infty}$ kernels on torus, including the Biot-Savart kernel. Feng and Wang [FW23] recently extended quantitative particle approximation results for the 2-dimensional Navier-Stokes equations to the whole space. More recently, Carrillo, Feng, Guo, Jabin and Wang [CFG⁺24] used the relative entropy method to study the particle approximation of the spatially homogeneous Landau equation for Maxwellian molecules. In [Lac23], Lacker developed a new local relative entropy method, achieving optimal quantitative estimates between $\mathcal{L}^{N,k}(t)$ and $v_t^{\otimes k}$. Recently, Wang [Wan24] extended this approach to handle particle systems with singular $W^{-1,\infty}$ kernels.

A key distinction of our model compared to classical particle systems lies in the introduction of environmental noise W, which induces stochasticity in the mean field limit $(v_t)_{t\in[0,T]}$. Specifically, the mean field limit $(v_t)_{t\in[0,T]}$ satisfies a stochastic nonlinear Fokker-Planck equation (1.5) with transport noise, rather than a deterministic partial differential equation. Moreover, $(v_t)_{t\in[0,T]}$ coincides with the conditional law of $\bar{X}_i(t)$, which is the solution to the following conditional McKean-Vlasov equation (1.7), with respect to the environmental noise $\{W_t, t\in[0,T]\}$, i.e., $v_t(\mathrm{d}x) = \mathcal{L}(\bar{X}_i(t)|\mathcal{F}_T^W)(\mathrm{d}x)$ in the sense of $\mathcal{P}(\mathbb{T}^2)$,[1]

$$\bar{X}_i(t) = X_i(0) + \int_0^t K * v_s(\bar{X}_i(s))\mathrm{d}s + \sqrt{2}B_i(t) + \int_0^t \sigma(\bar{X}_i(s)) \circ \mathrm{d}W_s. \tag{1.7}$$

This setup allows us to establish a *conditional propagation of chaos* type result. For fixed $k\in\mathbb{N}$, as the number of particles $N\to\infty$, we have

$$F^{N,k}(t) \rightharpoonup \bar{F}^{N,k}(t) := v_t^{\otimes k}.$$

Here $F^{N,k}(t)(\mathrm{d}x^N)$ is the k-marginal of the conditional distribution $F^N(t)(\mathrm{d}x^N)$ of $X^N(t)$ with respect to the environmental noise $\{W_t, t\in[0,T]\}$, defined by

$$\mathcal{L}(X^N(t)|\mathcal{F}_T^W)(\mathrm{d}x^N), \tag{1.8}$$

and $\bar{F}^{N,k}(t)(\mathrm{d}x^N)$ is indeed the k-marginal of the conditional distribution $\bar{F}^N(t)(\mathrm{d}x^N)$ of $\bar{X}^N(t)$ with respect to the environmental noise $\{W_t, t\in[0,T]\}$, defined by

$$\mathcal{L}(\bar{X}^N(t)|\mathcal{F}_T^W)(\mathrm{d}x^N). \tag{1.9}$$

We refer to [SZ24] for more details.

One motivation for considering a particle system with common noise is its ability to describe environmental influences on particles across various fields. For example, in mathematical finance, such a model reflects the fact that a large financial market should include a common set of assets accessible to all agents (see e.g., [Lac15]). Additionally, it can help us study the phenomenon known as regularization by noise (see e.g., [Fla11]). However, the literature on the mean field limit in the presence of environmental noise remains limited. We refer readers to the works [KX99, CF16b, Ros20, NRS21, SZ24, Nik24]. Relative entropy method has also been developed to address the case of environmental noise. In [SZ24], we derived a quantitative particle approximation for the stochastic 2-dimensional Navier-Stokes equation. In [Nik24], Nikolaev established the relative entropy method for particle systems with a general kernel $K\in L^\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ and Ito's noise on the whole space.

[1] We use $\mathcal{P}(\mathbb{T}^2)$ to denote the space of probability measures on $\mathbb{T}^2$.

Another interesting result is presented in [FL21, GL23], where it is shown that the mean field limit equation can still be a deterministic partial differential equation by rescaling the space covariance of the noise as the number of particles increases.

Recently, Wang, Zhao and Zhu [WZZ23] study the limiting behavior of the fluctuation process η^N for the interacting particle system (1.1) with $\sigma = 0$ on torus $\mathbb{T}^d$ by martingale method. They focused on systems with singular kernels satisfying $\|K\|_{L^\infty} < \infty$, or $K(x) = -K(-x)$ and $\|xK(x)\|_{L^\infty} < \infty$, which include important examples such as the Biot-Savart kernel. Using the Itô's formula, the fluctuation measure η^N can be formally represented by the following SPDE

$$\begin{aligned} d\eta_t^N =& \Delta\eta_t^N \mathrm{d}t - \nabla\cdot\big(vK*\eta^N + \eta^N K*v + \frac{1}{\sqrt{N}}\eta^N K*\eta^N\big)\mathrm{d}t \\ &+ \frac{1}{2}\sigma\cdot\nabla(\sigma\cdot\nabla\eta_t^N)\mathrm{d}t + \mathrm{d}\mathcal{M}_t^N - (\sigma\cdot\nabla\eta_t^N)\mathrm{d}W_t, \end{aligned} \tag{1.10}$$

where $\mathcal{M}_t^N$ is a continuous stochastic process taking values in $H^{-\alpha-1}(\mathbb{T}^2)$ for every $\alpha > 1$, introduced in Section 3.2 below. The most challenging part lies in studying the uniform estimates and convergence for the interacting term

$$vK*\eta^N + \eta^N K*v + \frac{1}{\sqrt{N}}\eta^N K*\eta^N,$$

due to the singularity of the kernel K and the fluctaution measures $(\eta^N)_{N\geqslant 1}$(which live in negative Sobolev spaces). By the relative entropy method, Wang, Zhao and Zhu [WZZ23] addressed the challenging issues of uniform estimates and the convergence of interaction term. They showed the convergence from the fluctaution measures $(\eta^N)_{N\geqslant 1}$ to the solution η of the fluctuation SPDE (1.11) with $\sigma = 0$ below. Building on the results of [WZZ23], this paper extends their framework to incorporate the presence of common noise. Specifically, we demonstrate that the sequence of fluctuation measures $(\eta^N)_{N\geqslant 1}$ for the stochastic point vortex model with common noise (1.1) converges in distribution to a stochastic process η in the space $L^2([0,T], H^{-\alpha}) \cap C([0,T], H^{-\alpha-2})$ for every $\alpha > 1$. Here η is the unique probabilistically strong solution to the following fluctuation SPDE (1.11)

$$d\eta_t = \Delta\eta_t \mathrm{d}t - \nabla\cdot(v_t K*\eta_t)\mathrm{d}t - \nabla\cdot(\eta_t K*v_t)\mathrm{d}t + \frac{1}{2}\sigma\cdot\nabla(\sigma\cdot\nabla\eta_t)\mathrm{d}t + \mathrm{d}\mathcal{M}_t - (\sigma\cdot\nabla\eta_t)\mathrm{d}W_t, \tag{1.11}$$

where $\{v_t, t\in[0,T]\}$ is the unique probabilistically strong solution to the stochastic 2-dimensional Navier-Stokes equation (1.5) with initial value $v_0 \in H^3$ with strictly positive lower bound, i.e., $\inf_{\mathbb{T}^2} v_0 > 0$. Compared to the case without environmental noise W, the final fluctuation equation (1.11) includes two noise terms. The first is a multiplicative transport noise $\sigma\cdot\nabla\eta_t \mathrm{d}W_t$, which arises from the environmental noise W. The second is an additive noise $\mathcal{M}_t$. As shown in [WZZ23], $\{\mathcal{M}_t, t\in[0,T]\}$ is a continuous Gaussian process taking values in $H^{-\alpha-1}(\mathbb{T}^2)$ for every $\alpha > 1$, in the absence of environmental noise W. In the presence of environmental noise, it remains a continuous stochastic process taking values in $H^{-\alpha-1}(\mathbb{T}^2)$ for every $\alpha > 1$, but it is no longer Gaussian. Instead, its conditional distribution satisfies

$$\begin{aligned} \mathbb{E}\Big[\exp i\,\langle\varphi, \mathcal{M}_t\rangle \mid \mathcal{F}_T^W\Big] &= \exp\Big\{-\int_0^t \langle|\nabla\varphi|^2, v_s\rangle \mathrm{d}s\Big\}, \\ \mathbb{E}\Big[\exp i\,\langle\varphi, (\mathcal{M}_{t+r} - \mathcal{M}_t)\rangle \mid \mathcal{F}_T^W \vee \mathcal{F}_t^{\mathcal{M}}\Big] &= \exp\Big\{-\int_t^{t+r} \langle|\nabla\varphi|^2, v_s\rangle \mathrm{d}s\Big\}, \end{aligned} \tag{1.12}$$

for every $\varphi \in C^\infty(\mathbb{T}^2), 0 \leqslant t < t+r \leqslant T$, where $(\mathcal{F}_t^W)_{t\in[0,T]}$ is the normal filtration generated by environmental noise W and $(\mathcal{F}_t^{\mathcal{M}})_{t\in[0,T]}$ is the normal filtration generated by additive noise $\mathcal{M}$. Therefore, it can be inferred that under the influence of environmental noise W, the distribution of η is no longer Gaussian. Further explanations on the conditional distribution of $\mathcal{M}_t$ can be found in Remark 2.5.

Theorem 1.1. *Assume that the identical distribution $\mathcal{L}(X(0))$ on torus $\mathbb{T}^2$ for the i.i.d. initial values $\{X_i^N(0), i\in\mathbb{N}\}$ has a density $v_0 \in H^3(\mathbb{T}^2)$ with strictly positive lower bound, i.e.,* $\inf_{\mathbb{T}^2} v_0 >$

0. *the sequence of fluctuation measures* η^N *for the stochastic point vortex model* (1.1) *converges in distribution to* η *in the space*

$$L^2([0,T],H^{-\alpha})\cap C([0,T],H^{-\alpha-2}),$$

for every $\alpha>1$, *where* η *is the unique probabilistically strong solution to the fluctuation SPDE* (1.11) *in the sense of Definitions 2.6 and 2.7 below.*

The definition of the solutions to the fluctuation SPDE (1.11) can be found in Section 2, Definitions 2.6 and 2.7. During this process, we also obtain the well-posedness of the fluctuation SPDE (1.11).

Corollary 1.2. *Given a 1-dimensional Brownian motion* $\{W_t,t\in[0,T]\}$ *and a continuous stochastic process* $\{\mathcal{M}_t,t\in[0,T]\}$ *taking values in* $H^{-\alpha-1}(\mathbb{T}^2)$ *for every* $\alpha>1$, *on probability space* $(\Omega,\mathcal{F},\mathbb{P})$, *satisfying* (1.12), *for each* $\eta_0\in H^{-\alpha},\forall\alpha>1,\mathbb{P}$*-a.s., satisfying for every* $\varphi\in C^\infty(\mathbb{T}^2)$,

$$\mathcal{L}(\langle\eta_0,\varphi\rangle)=\mathcal{N}\Big(0,\langle\varphi^2,v_0\rangle-\langle\varphi,v_0\rangle^2\Big),$$

there exists a unique probabilistically strong solution to (1.11). *Here* $\mathcal{L}(\langle\eta_0,\varphi\rangle)$ *denotes the distribution of* $\langle\eta_0,\varphi\rangle$, *and* $\mathcal{N}(0,a)$ *denotes the centered Gaussian distribution on* $\mathbb{R}$ *with variance* a.

Similar to the mean-field limit result (1.6), the central limit theorem for the interacting particle system (1.1) reflects the asymptotic independence of particles. Specifically, in the absence of environmental noise W, the distribution of the fluctuation measures $\{\eta^N\}_{N\in\mathbb{N}}$ becomes asymptotically Gaussian as $N\to\infty$, like i.i.d. random variables. In the presence of environmental noise W, our result shows that the asymptotic independence is reflected in an additive noise $\mathcal{M}_t$, which from (1.12) is "Gaussian conditioned on $\mathcal{F}_T^W$." Additionally, the effect of environmental noise W is captured in the conditional distribution of $\mathcal{M}_t$ and the transport noise $\sigma\cdot\nabla\eta_t\mathrm{d}W_t$, analogous to the mean-field limit equation (1.5).

1.1. **Related literatures.** For the fluctuations of interacting diffusions, which is the focus of this article, one of the earliest results is due to Itô [Itô83], where he showed that for the system of 1-dimensional independent and identically distributed Brownian motions, the limit of the corresponding fluctuations is a Gaussian process. One common method to study the central limit theorem for interacting particle systems (1.1) is the martingale method, which is also employed in this paper. A significant contribution in this area was made by Fernandez and Méléard [FM97], who studied the fluctuation behaviour of particle systems with well-regularized kernels and multiplicative independent noise on the whole space. It was shown that the fluctuation process η^N, as a weighted Sobolev space-valued random variable, converges to a Gaussian-distributed limit in the sense of distribution, as $N\to\infty$. The requirement for kernels with strong regularity arises for two main reasons. First, the uniform estimates needed to prove tightness rely on a coupling method, which requires at least Lipschitz regularity for the kernels. Second, when identifying the tight limit, the second-order differential operator in the fluctuation equation (1.11) must be linear continuous in the weighted Sobolev space, necessitating stricter regularity conditions. It is worth emphasizing that the approach introduced in [FM97] has been amplified to study various interacting models, see [JM98, Che17, CF16a, LS16] etc. Recent work by Wang, Zhao and Zhu [WZZ23] extends the results to the limiting behavior of the fluctuation process η^N for interacting particle system (1.1) on torus with singular kernel satisfying $\|K\|_{L^\infty}<\infty$, or $K(x)=-K(-x)$ and $\|xK(x)\|_{L^\infty}<\infty$ (e.g., Biot-Savart kernel). Using the relative entropy method and a structured observation of interaction term, they addressed the challenging issues of uniform estimates and the convergence of interaction terms when applying the martingale method to particle systems with singular kernels. Building on [WZZ23], our work applies their framework to resolve singularity issues in the Biot–Savart law. Another study closely related to our work is [KX04], which employed a martingale method combined with coupling method to investigate the convergence of the fluctuation process for interacting particle systems with common noise and time-varying random intensities $(\xi_j)_{j\geqslant1}$ in the modified Schwartz space. In their model, the random intensities ξ_j are governed by a stochastic differential equation driven by independent noise B_j and the environmental noise W. The limiting distribution of the fluctuation process is non-Gaussian, consistent with our findings. Similar to [FM97], their method also required strong regularity assumptions on the interaction kernel K. More recently, Pengzhi Xie [Xie24] studied the quantitative central limit theorem for particle systems with summable Fourier modes kernels. We also refer to the works [Oel87, JM98, CHJ24] which study the fluctuations in the moderate mean field regime.

Additionally, another type result is known as the pathwise central limit theorem. This kind of result studies the limiting behavior of fluctuation processes based on particle trajectories, considering the entire paths of particles. For example, we consider the fluctuation processes of the form $\sqrt{N}(\frac{1}{N}\sum_{i=1}^{N}\delta_{X_i} - \mathcal{L}(X))$, where $X \in C([0,T],\mathbb{R}^d)$ solves some nonlinear stochastic differential equation. In this context, Tanaka and Hitsuda [TH81] first studied a specific case with $K(x) = -\lambda x, \lambda > 0$. Later, Tanaka [Tan84] extended the analysis to more general kernels $K \in C_b^2$, using a pathwise construction approach. Sznitmann [Szn84] removed the differentiability condition on test functions and generalized the result to bounded and Lipschitz continuous kernel, using Girsanov's formula and the method of U-statistics. Recently, Budhiraja and Wu [BW16] studied some general interacting systems with possible common factors, which do not necessarily have the exchangability property as usual. Their result follows the strategy by Sznitmann [Szn84]. More recently, Chaintron [Cha24] further generalized the results to include the case of multiplicative independent noise.

The qualitative central limit theorem for second-order systems is explored in [BH77] and [Lan09]. Among these, [BH77] as the first to investigate the fluctuation behavior of second-order systems. More recently, significant advancements have been made in [Due21] and [BD24]. Duerinckx [Due21] achieved optimal quantitative fluctuation estimates, while Bernou and Duerinckx [BD24] established an uniform-in-time quantitative central limit theorem.

1.2. **Outline of proof and difficulties.** The proof in this paper consists of two main steps: Step 1 involves establishing tightness, and Step 2 focuses on identifying the tight limit and proving the pathwise uniqueness of the fluctuation SPDE (1.11).

In the first step, we start by obtaining the necessary uniform estimates for the fluctuation process $(\eta^N)_{N\in\mathbb{N}}$ and the interacting terms $\nabla\cdot[K*\mu_N(t)\mu_N(t) - v_t K * v_t]$, $\mathcal{K}^N(\varphi)$ defined in (3.2) below, using the relative entropy method. The main challenge in this step is to establish the tightness of the laws of $\{\eta^N\}$ based on these estimates. On one hand, the uniform bound for the relative entropy $\sup_{t\in[0,T]} H(F_t^N|\bar{F}_t^N)(\omega)$ depends on $\omega \in \Omega$, due to the singularity of the Biot–Savart kernel and the presence of environmental noise $\{W_t, t \in [0,T]\}$. This prevents us from obtaining the tightness of the laws of $\{\eta^N\}$ directly. However, when K is bounded or there is no common noise, the uniform relative entropy $\sup_{t\in[0,T]} H(F_t^N|\bar{F}_t^N)$ can be bounded by a deterministic constant. On the other hand, compared to the case without common noise in [WZZ23], we must handle the uniform estimate for the Hölder seminorm of the new transport noise term $\sigma\cdot\nabla\eta_t^N \mathrm{d}W_t$. To address these challenges, based on [WZZ23], a classical localization method is applied. Specifically, we study tightness through introducing two sequence of stopping times to control the mean field limit v and the fluctuation measures η^N.

In the second step, we identify the tight limit $\tilde{\eta}$ in Proposition 3.9 and establish the existence of solutions to the fluctaution SPDE (1.11). The main difficulty in this step is identifying the additive noise $\{\mathcal{M}_t, t \in [0,T]\}$ in the fluctution SPDE (1.11) through studying the conditional law of $\{\mathcal{M}_t, t \in [0,T]\}$ with respect to the environmental noise $\{W_t, t \in [0,T]\}$. To overcome this challenge, we establish a strong convergence (see Proposition 4.1 below) from the stochastic point vortex system (1.1) to the conditional Mckean-Vlasov equation (1.7), and then derive the conditional law of the additive noise $\{\mathcal{M}_t, t \in [0,T]\}$, following the idea in [KX04]. Pathwise uniqueness for the fluctuation SPDE (1.11) is then proven using standard SPDE arguments.

Organization of the paper. The paper is structured as follows. Section 2 provides an introduction to key definitions related to the mean field limit equation (1.5), the fluctuation SPDE (1.11), and some auxiliary results that will be used in the subsequent proofs. The main sections of the paper are

Section 3 and Section 4. In Section 3, we study the tightness of laws of $\{\eta^N\}_{N\in\mathbb{N}}$ in the space $\mathcal{X}$ (defined in Definition (2.4)) and $\{\mathcal{M}^N\}_{N\in\mathbb{N}}$ in the space $\mathcal{Y}$ (defined in Definition (2.4)). In Section 4, we establish the well-posedness of the fluctuation SPDE (1.11) and convergence of the fluctuation process $\{\eta^N\}_{N\in\mathbb{N}}$ to the unique probabilistically strong solution η to the fluctuation SPDE (1.11) in the sense of distribution.

At the end of this section, we introduce the basic notation used throughout the paper.

(1) Bracket notations: The bracket $\langle\cdot,\cdot\rangle$ denotes integrals when the space and underlying measure are clear from the context. We use a similar bracket $[\cdot,\cdot]_t$ to denote quadratic variations between local martingales at time t.
(2) Filtration notations: The notation $\mathcal{F}_1\vee\mathcal{F}_2$ stands for the σ-algebra generated by $\mathcal{F}_1\cup\mathcal{F}_2$. Given a stochastic process $X_t, t\in[0,T]$, we use $(\mathcal{F}_t^X)_{t\in[0,T]}$ to denote the normal filtration generated by X. In particular, we use $(\mathcal{F}_t^W)_{t\in[0,T]}$ to denote the normal filtration generated by 1-dimensional Brownian motions $\{W_t, t\in[0,T]\}$. We also use $(\mathcal{F}_t)_{t\in[0,T]}$ to denote the normal filtration generated by 1-dimensional Brownian motion W and 2-dimensional Brownian motions $(B_i)_{i\geqslant 1}$ and $(X_i(0))_{i\geqslant 1}$.
(3) Distribution notations: Given a Polish space E and probability space $(\Omega,\mathcal{F},\mathbb{P})$, we say $Q(\mathrm{d}x,\omega)$ is a random meausre on E, if $Q(\mathrm{d}x,\omega)$ is a function of two variables $\omega\in\Omega$ and $A\in\mathbb{B}(E)$, satisfies that there exists a null set $N\in\mathcal{F}$ such that $Q(\mathrm{d}x,\omega)$ is a measure in A for fixed $\omega\in N^c$. Given σ-algebra $\mathcal{F}$, we use $\mathcal{L}(X)$, $\mathcal{L}(X|\mathcal{F})$ to denote the distribution of X and conditional distribution of X with respect to $\mathcal{F}$. In particular, for convention, we may denote the distribution by its density function when the distribution has a density function. Given a symmetric probability measure ρ^N on E^N where E is a Polish space, the k-marginal $\rho^{N,k}$ is a probability measure on E^k defined by $\int_{E^{N-k}}\rho^N(\mathrm{d}x_1\cdots\mathrm{d}x_N)$, where $k\leqslant N$. Finally, we use $\mathcal{P}(E)$ to denote the space of probability measure on E.
(4) Independence and Conditional independence: Given three σ-algebras $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$, we use $\mathcal{F}_1\perp\mathcal{F}_2|\mathcal{F}_3$ to indicate that $\mathcal{F}_1$ and $\mathcal{F}_2$ are conditionally independent given $\mathcal{F}_3$. $\mathcal{F}_1\perp\mathcal{F}_2$ indicates that $\mathcal{F}_1$ and $\mathcal{F}_2$ are mutually independent.
(5) Product space and Product function: Given two measure spaces $(\Omega_1,\mathcal{A}_1,\mu_1)$ and $(\Omega_2,\mathcal{A}_2,\mu_2)$, we denote their product space as $\Omega_1\times\Omega_2$, the product σ-algebra as $\mathcal{A}_1\times\mathcal{A}_2$, and the product measure as $\mu_1\times\mu_2$. For $k\in\mathbb{N}$, we use $(\Omega_1^{\otimes k},\mathcal{A}_1^{\otimes k},\mu_1^{\otimes k})$ to denote the k-product measure space for the measure space $(\Omega_1,\mathcal{A}_1,\mu_1)$. Given a function $f(x), x\in E$ on Polish space E and $k\in\mathbb{N}$, the k-tensorized function $f^{\otimes k}(x^k)$ is defined by $\prod_{i=1}^k f(x_i)$, where $x^k=(x_1,\cdots,x_k)\in E^k$.
(6) We will mostly work on Sobolev spaces. The norm of Sobolev space $H^\alpha(\mathbb{T}^d)$, $\alpha\in\mathbb{R}$, is defined by
$$\|f\|_{H^\alpha}^2 := \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^\alpha|\langle f,e_k\rangle|^2,$$
where $e_k := e^{\sqrt{-1}k\cdot x}, k\in\mathbb{Z}^d$. We also use some results on Besov spaces $B_{p,q}^\alpha$ in this paper and provide a brief introduction about Besov spaces in Section 2.3.

Finally, throughout this paper, we use C to denote universal constants, and we indicate relevant dependencies using subscripts when necessary. We use the notation $a\lesssim b$ if there exists a universal constant $C>0$ such that $a\leqslant Cb$.

2. Preliminaries

In this section, we introduce the definitions of solutions and collect some auxiliary results.

2.1. **Definitions of solutions.** In this subsection, we present several distinct definitions for solutions to the stochastic 2-dimensional Navier-Stokes equation (1.5) and the fluctuation SPDE (1.11). The following definitions for solutions to the stochastic 2-dimensional Navier-Stokes equation (1.5) are consistent with those in [SZ24] and the well-posedness of (1.5) has also been established in [SZ24].

Definition 2.1. *A probabilistically weak solution* $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P}, W, (v_t)_{t\in[0,T]})$ *to* (1.5) *with initial value* $v_0 \in H^3(\mathbb{T}^2)$ *is defined as a stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$ *supporting standard* $(\mathcal{G}_t)_{t\in[0,T]}$*-Brownian motion* $\{W_t, t \in [0,T]\}$ *(denoted by* W*) and a continuous* $L^2(\mathbb{T}^2)$*-valued* $\mathcal{G}_t$*-adapted stochastic process* $(v_t)_{t\in[0,T]}$ *such that*

(1) *For all* $t \in [0,T]$,
$$ess\sup_{x\in\mathbb{T}^2} v_t \leqslant ess\sup_{x\in\mathbb{T}^2} v_0, \quad ess\inf_{x\in\mathbb{T}^2} v_t \geqslant ess\inf_{x\in\mathbb{T}^2} v_0, \quad \|v_t\|_{L^2} \leqslant \|v_0\|_{L^2}, \quad \mathbb{P}-a.s.. \tag{2.1}$$

(2) *It holds that*
$$\mathbb{E}\int_0^T \|v_t\|_{H^4}^2 \mathrm{d}t < \infty, \quad \mathbb{E}\big[\sup_{t\in[0,T]} \|v_t\|_{H^2}^2\big] < \infty. \tag{2.2}$$

(3) *For all* $\varphi \in C^\infty(\mathbb{T}^2)$*, it holds almost surely that for all* $t \in [0,T]$,
$$\langle \varphi, v_t\rangle = \langle \varphi, v_0\rangle + \int_0^t \langle \Delta\varphi, v_s\rangle \,\mathrm{d}s + \int_0^t \langle \nabla\varphi, K * v_s v_s\rangle \,\mathrm{d}s + \int_0^t \langle \nabla\varphi, v_s\sigma\rangle \circ \mathrm{d}W_s.$$

Definition 2.2. *Given a independent 1-dimensional Brownian motion* $\{W_t, t\in[0,T]\}$ *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, we say that* $(v_t)_{t\in[0,T]}$ *is a probabilistically strong solution to* (1.5) *with initial value* $v_0 \in H^3(\mathbb{T}^2)$ *if* $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t\in[0,T]}, \mathbb{P}, (v_t)_{t\in[0,T]})$ *is a probabilistically weak solution to* (1.5) *with initial value* $v_0 \in H^3(\mathbb{T}^2)$*, where* $(\mathcal{F}_t^W)_{t\in[0,T]}$ *is the normal filtration generated by Brownian motions* $\{W_t, t \in [0,T]\}$.

Definition 2.3. *We say that pathwise uniqueness holds for* (1.5) *if for any two probabilistically weak solutions* $(v_t)_{t\in[0,T]}$ *and* $(\tilde{v}_t)_{t\in[0,T]}$ *on the same stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$*, with the same noise* $\{W_t, t\in[0,T]\}$ *and the same initial data* v_0*, it satisfies*
$$\mathbb{P}\Big(\|v_t - \tilde{v}_t\|_{L^2} = 0, \forall t \in [0,T]\Big) = 1.$$

Before introducing the definitions of solutions to (1.11), we first define a Polish space in which the solution exists. This space is given by
$$\mathcal{D} := \mathcal{X} \times \mathcal{Y} \times \mathcal{W} \tag{2.3}$$
equipped with the metric $d_{\mathcal{D}}(f,g) := (\sum_{i=\mathcal{X},\mathcal{Y},\mathcal{W}} d_i^2(f,g))^{\frac{1}{2}}$, where
$$\begin{aligned}
\mathcal{X} &:= \bigcap_{k\in\mathbb{N}} \Big[C([0,T]; H^{-3-\frac{1}{k}}(\mathbb{T}^2)) \cap L^2([0,T]; H^{-1-\frac{1}{k}}(\mathbb{T}^2))\Big],\\
\mathcal{Y} &:= \bigcap_{k\in\mathbb{N}} C([0,T]; H^{-2-\frac{1}{k}}(\mathbb{T}^2)),\\
\mathcal{W} &:= C([0,T];\mathbb{R}),
\end{aligned}$$
endowed with the metrics
$$\begin{aligned}
d_{\mathcal{X}}(f,g) &:= \sum_{k=1}^\infty 2^{-k}\Big(1 \wedge \Big(\|f-g\|_{C([0,T];H^{-3-\frac{1}{k}})} + \|f-g\|_{L^2([0,T];H^{-1-\frac{1}{k}})}\Big)\Big),\\
d_{\mathcal{Y}}(f,g) &:= \sum_{k=1}^\infty 2^{-k}\big(1 \wedge \|f-g\|_{C([0,T];H^{-2-\frac{1}{k}})}\big).
\end{aligned}$$
Similarly, we define Polish space
$$\mathcal{H} := \mathcal{V} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{W} \tag{2.4}$$
equipped with the metric $d_{\mathcal{H}}(f,g) := (\sum_{i=\mathcal{V},\mathcal{X},\mathcal{Y},\mathcal{W}} d_i^2(f,g))^{\frac{1}{2}}$, where
$$\mathcal{V} := C([0,T]; L^2(\mathbb{T}^2)) \cap L^2([0,T]; H^4(\mathbb{T}^2))$$

endowed with the metric $d_{\mathcal{V}}(f,g) := \|f-g\|_{C([0,T];L^2)} + \|f-g\|_{L^2([0,T];H^4)}$. We then give the definitions about the solution to the fluctuation SPDE (1.11).

Definition 2.4. *A probabilistically weak solution* $\Big(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P}, \Big(\eta_t, \mathcal{M}_t, W_t\Big)_{t\in[0,T]}\Big)$ *to the SPDE* (1.11) *is defined as a stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$ *supporting the stochastic process*

$$\Big(\eta_t, \mathcal{M}_t, W_t\Big)_{t\in[0,T]}$$

valued in $\mathcal{D}$,

(1) W *is* $(\mathcal{G}_t)_{t\in[0,T]}$*-1-dimensional Brownian motion.*
(2) $(\mathcal{M}_t)_{t\in[0,T]}$ *is a* $\mathcal{G}_t$*-adapted process belonging to* $C([0,T]; H^{-\alpha}(\mathbb{T}^2))$ $\mathbb{P}$*-a.s., for every* $\alpha > 2$*, and satisfying for every* $\varphi \in C^\infty(\mathbb{T}^2), 0 \leqslant t < t + r \leqslant T$,

$$\mathbb{E}\Big[\exp i\,\langle\varphi, \mathcal{M}_t\rangle \mid \mathcal{F}_T^W\Big] = \exp\Big\{-\int_0^t \langle |\,\nabla\varphi\,|^2, v_s\rangle \mathrm{d}s\Big\},$$
$$\mathbb{E}\Big[\exp i\,\langle\varphi, (\mathcal{M}_{t+r} - \mathcal{M}_t)\rangle \mid \mathcal{F}_T^W \vee \mathcal{F}_t^{\mathcal{M}}\Big] = \exp\Big\{-\int_t^{t+r} \langle |\,\nabla\varphi\,|^2, v_s\rangle \mathrm{d}s\Big\},$$

(3) $(\eta_t)_{t\in[0,T]}$ *is a continuous* $H^{-\alpha-2}(\mathbb{T}^2)$*-valued* $\mathcal{G}_t$*-adapted stochastic process satisfying* $\eta \in L^2([0,T], H^{-\alpha}(\mathbb{T}^2))$ $\mathbb{P}$*-a.s., for every* $\alpha > 1$.
(4) *For all* $\varphi \in C^\infty(\mathbb{T}^2)$*, it holds almost surely that for all* $t \in [0,T]$,

$$\begin{aligned}\langle\eta_t, \varphi\rangle =& \langle\eta_0, \varphi\rangle + \int_0^t \langle\Delta\varphi, \eta_s\rangle \mathrm{d}s + \int_0^t \langle\nabla\varphi, v_s K * \eta_s\rangle \mathrm{d}s + \int_0^t \langle\nabla\varphi, \eta_s K * v_s\rangle \mathrm{d}s \\ &+ \langle\mathcal{M}_t, \varphi\rangle + \frac{1}{2}\int_0^t \Big\langle \sigma\cdot\nabla\Big(\sigma\cdot\nabla\varphi\Big), \eta_s\Big\rangle \mathrm{d}s + \int_0^t \langle\sigma\cdot\nabla\varphi, \eta_s\rangle \,\mathrm{d}W_s,\end{aligned} \tag{2.5}$$

where $(v_t)_{t\in[0,T]}$ *is the unique probabilistically strong solution to the mean field equation* (1.5) *in the sense of Definitions 2.2 and 2.3.*

Remark 2.5. *Condition (2) in Definition 2.4 specifies the conditional distribution of* $\mathcal{M}$ *with respect to the environmental noise* W*, which uniquely determines the joint distribution of* $\mathcal{M}$ *and* W*. Notably, this condition also shows that given the environmental noise information* $\mathcal{F}_T^W$*, the distribution of* $\mathcal{M}_t$ *is similar to a Gaussian distribution.*

Definition 2.6. *Given a 1-dimensional Brownian motion* $\{W_t, t \in [0,T]\}$ *and a stochastic process* $\{\mathcal{M}_t, t \in [0,T]\}$ *with values in* $\bigcap_{k\in\mathbb{N}} C([0,T]; H^{-2-\frac{1}{k}}(\mathbb{T}^2))$ *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *satisfying for every* $\varphi \in C^\infty(\mathbb{T}^2), 0 \leqslant t < t + r \leqslant T$,

$$\mathbb{E}\Big[\exp i\,\langle\varphi, \mathcal{M}_t\rangle \mid \mathcal{F}_T^W\Big] = \exp\Big\{-\int_0^t \langle |\,\nabla\varphi\,|^2, v_s\rangle \mathrm{d}s\Big\},$$
$$\mathbb{E}\Big[\exp i\,\langle\varphi, (\mathcal{M}_{t+r} - \mathcal{M}_t)\rangle \mid \mathcal{F}_T^W \vee \mathcal{F}_t^{\mathcal{M}}\Big] = \exp\Big\{-\int_t^{t+r} \langle |\,\nabla\varphi\,|^2, v_s\rangle \mathrm{d}s\Big\},$$

we say that $(\eta_t)_{t\in[0,T]}$ *is a probabilistically strong solution to* (1.11) *if*

$$\Big(\Omega, \mathcal{F}, (\mathcal{F}_t^{W,\mathcal{M}})_{t\in[0,T]}, \mathbb{P}, \Big(\eta_t, \mathcal{M}_t, W_t\Big)_{t\in[0,T]}\Big)$$

is a probabilistically weak solution to (1.11) *where* $(\mathcal{F}_t^{W,\mathcal{M}})_{t\in[0,T]}$ *is the normal filtration generated by* $\{W_t, t \in [0,T]\}$, $\{\mathcal{M}_t, t \in [0,T]\}$*, and the initial value* η_0.

Definition 2.7. *We say that pathwise uniqueness holds for* (1.11) *if for any two probabilistically weak solutions* $(\eta_t)_{t\in[0,T]}$ *and* $(\tilde{\eta}_t)_{t\in[0,T]}$ *on the same stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$*, with the* *same noise* $\{W_t, t\in[0,T]\}, \{\mathcal{M}_t, t\in[0,T]\}$ *and the same initial data* $\eta_0 \in \cap_{k\in\mathbb{N}} H^{-1-\frac{1}{k}}(\mathbb{T}^2)$*, it satisfies that for every* $4>\alpha>3$,

$$\mathbb{P}(\sup_{t\in[0,T]} \|\eta_t - \tilde{\eta}_t\|^2_{H^{-\alpha}} = 0) = 1.$$

2.2. **Relative entropy and stochastic point vortex model.** In this section, we collect the auxiliary results from [JW18], [SZ24] and [FHM14] for convenience. We start this section by recalling the definition of relative entropy associated to any two probability measures ρ and η on Polish space E.

The *relative entropy* $H(\rho|\eta)$ is defined as

$$H(\rho|\eta) := \begin{cases} \int_E \log \frac{d\rho}{d\eta} \mathrm{d}\rho & \rho \ll \eta; \\ +\infty, & otherwise. \end{cases}$$

Here $\frac{d\rho}{d\eta}$ represents the Radon–Nikodym derivative of ρ with respect to η.

The following lemma is used to derive the uniform estimates for the stochastic point vortex model (1.1), i.e., Lemma 3.1, in Section 3.

Lemma 2.8 ([JW18, Lemma 1])**.** *For any two probability densities* $\rho_N, \bar{\rho}_N$ *on* $\mathbb{T}^{2N}, N\geqslant 1$ *and any function* $\phi\in L^\infty(\mathbb{T}^{2N})$*, one has that for any constant* $b>0$,

$$\int_{\mathbb{T}^{2N}} \phi\rho_N \mathrm{d}x^N \leqslant \frac{1}{bN}\left(H(\rho_N|\bar{\rho}_N) + \log\int_{\mathbb{T}^{2N}} \bar{\rho}_N \exp\{bN\phi\}\mathrm{d}x^N\right).$$

The following two results from [SZ24] establish the well-posedness of the stochastic 2-dimensional Navier-Stokes equation (1.5) and provide a quantitative conditional propagation of chaos result for the stochastic point vortex model (1.1).

Lemma 2.9 ([SZ24, Theorem 3.1])**.** *Given 1-dimensional standard Brownian motion* $\{W_t, t\in[0,T]\}$ *on probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, for each* $v_0\in H^3(\mathbb{T}^2)$*, there exists a unique probabilistically strong solution* $(v_t)_{t\in[0,T]}$ *to* (1.5) *in the sense of Definitions 2.2 and 2.3.*

Lemma 2.10 ([SZ24, Theorem 1.1])**.** *Assume that the probability measure* $v_0 = \mathcal{L}(X(0))$ *on* $\mathbb{T}^2$ *has a density* $v_0\in H^3(\mathbb{T}^2)$ *and* $\inf_{x\in\mathbb{T}^2} v_0 > 0$*. Then, it holds that*

$$H(F_t^N|\bar{F}_t^N) \leqslant \exp\left(C_0\int_0^t (\|v_s\|^2_{H^4}+1)\mathrm{d}s\right) \quad \forall t\in[0,T] \quad \mathbb{P}-a.s.,$$

where C_0 *is a positive deterministic constant depending on* $\|v_0\|_{L^2(\mathbb{T}^2)}$ *and* $\inf_{x\in\mathbb{T}^2} v_0$*,* F_t^N *and* $\bar{F}_t^N$ *are random measures on torus* $\mathbb{T}^{2N}$ *defined in* (1.8) *and* (1.9)*, and* $(v_t)_{t\in[0,T]}$ *is the unique probabilistically strong solution to the stochastic 2-dimensional Navier-Stokes equation* (1.5) *with the initial data* v_0 *in the sense of Definition 2.2 and 2.3.*

The following lemma will be applied in the proof of Proposition 4.1, in which the strong convergence between the stochastic point vortex model (1.1) and the conditional Mckean-Vlasov equation (1.7) is established.

Lemma 2.11 ([FHM14, Lemma 3.3])**.** *For any* $r\in(0,2)$ *and* $\beta>\frac{r}{2}$*, there exists a constant* $C_{r,\beta}>0$ *depends on* r,β *such that for any probability measure* ρ *on* $\mathbb{T}^2\times\mathbb{T}^2$ *with finite Fisher information* $I(\rho)$*,*

$$\int_{\mathbb{T}^2} \frac{1}{|x_1-x_2|^r}\rho(\mathrm{d}x_1\mathrm{d}x_2) \leqslant C_{r,\beta}(I^\beta(\rho)+1).$$

Here the Fisher information $I(\rho)$ *on* $\mathbb{T}^2$ *is defined by* $\int_{\mathbb{T}^2}\frac{|\nabla\rho|^2}{\rho}\mathrm{d}x$.

2.3. **Besov space.** In this subsection, we collect useful results related to Besov spaces. We use $(\Delta_i)_{i\geqslant -1}$ to denote the Littlewood-Paley blocks for a dyadic partition of unity. Besov spaces $B^\alpha_{p,q}$ on the torus with $\alpha\in\mathbb{R}$ and $1\leqslant p,q\leqslant\infty$, are defined as the completion of C^∞ with respect to the norm

$$\|f\|_{B^\alpha_{p,q}} := \Big(\sum_{n\geqslant -1}(2^{n\alpha q}\|\Delta_n f\|^q_{L^p})\Big)^{\frac{1}{q}}.$$

We remark that $B^\alpha_{2,2}$ coincides with the Sobolev space H^α, $\alpha\in\mathbb{R}$. We say $f\in C^\alpha$, $\alpha\in\mathbb{N}$, if f is α-times differentiable. For $\alpha\in\mathbb{R}\setminus\mathbb{N}$, we set $C^\alpha = B^\alpha_{\infty,\infty}$. We will often write $\|\cdot\|_{C^\alpha}$ instead of $\|\cdot\|_{B^\alpha_{\infty,\infty}}$. In the case $\alpha\in\mathbb{R}^+\setminus\mathbb{N}$, C^α coincides with the usual Hölder space. We use C^∞ to denote the space of infinitely differentiable functions on $\mathbb{T}^2$.

We quote the following results about Besov spaces.

Lemma 2.12 ([Tri06, Proposition 4.6])**.** *Let $\alpha\in\mathbb{R}$, $\beta\in\mathbb{R}$ and $p_1,p_2,q_1,q_2\in[1,\infty]$. Then the embedding*

$$B^\alpha_{p_1,q_2}\hookrightarrow B^\beta_{p_2,q_2}$$

is compact if and only if,

$$\alpha-\beta > d\Big(\frac{1}{p_1}-\frac{1}{p_2}\Big)_+.$$

Lemma 2.13. *(i) Let $\alpha,\beta\in\mathbb{R}$ and $p,p_1,p_2,q\in[1,\infty]$ be such that $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. The bilinear map $(u,v)\mapsto uv$ extends to a continuous map from $B^\alpha_{p_1,q}\times B^\beta_{p_2,q}$ to $B^{\alpha\wedge\beta}_{p,q}$ if $\alpha+\beta>0$ (cf.* [MW17, Corollary 2]*).*

(ii) (Duality.) Let $\alpha\in(0,1)$, $p,q\in[1,\infty]$, p' and q' be their conjugate exponents, respectively. Then the mapping $(u,v)\mapsto\langle u,v\rangle=\int uv\mathrm{d}x$ extends to a continuous bilinear form on $B^\alpha_{p,q}\times B^{-\alpha}_{p',q'}$, and one has $|\langle u,v\rangle|\lesssim\|u\|_{B^\alpha_{p,q}}\|v\|_{B^{-\alpha}_{p',q'}}$ (cf. [MW17, Proposition 7]*).*

Lemma 2.14 ([BCD11, Corollary 2.86])**.** *For any positive real number α and any $p,q\in[1,\infty]$, it holds that*

$$\|fg\|_{B^\alpha_{p,q}}\lesssim\|f\|_{L^\infty}\|g\|_{B^\alpha_{p,q}}+\|f\|_{B^\alpha_{p,q}}\|g\|_{L^\infty},$$

with the proportional constant independent of f and g.

Lemma 2.15 ([KS21, Theorem 2.1 and 2.2])**.** *Let $\alpha,\beta\in\mathbb{R}, q,q_1,q_2\in(0,\infty]$ and $p,p_1,p_2\in[1,\infty]$ be such that*

$$1+\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2},\quad \frac{1}{q}\leqslant\frac{1}{q_1}+\frac{1}{q_2}.$$

(1) If $f\in B^\alpha_{p_1,q}$ and $g\in L^{p_2}$, then $f*g\in B^\alpha_{p,q}$ and

$$\|f*g\|_{B^\alpha_{p,q}}\lesssim\|f\|_{B^\alpha_{p_1,q}}\cdot\|g\|_{L^{p_2}},$$

(2) If $f\in B^\alpha_{p_1,q_1}$ and $g\in B^\beta_{p_2,q_2}$, then $f*g\in B^{\alpha+\beta}_{p,q}$ and

$$\|f*g\|_{B^{\alpha+\beta}_{p,q}}\lesssim\|f\|_{B^\alpha_{p_1,q_1}}\cdot\|g\|_{B^\beta_{p_2,q_2}},$$

with the proportional constant independent of f and g.

3. Uniform estimates

The goal of this section is to prove the tightness of laws of the fluctuation measures $(\eta^N)_{N\in\mathbb{N}}$ and the tightness of laws of the additive noise $(\mathcal{M}^N)_{N\in\mathbb{N}}$. Recall that for $t\in[0,T]$, $\eta^N_t=\sqrt{N}(\mu_N(t)-v_t)$ and $\mu_N(t)=\frac{1}{N}\sum_{i=1}^N\delta_{X_i(t)}$. The additive noise term $(\mathcal{M}^N)_{N\in\mathbb{N}}$ given in Lemma 3.4 below, satisfies for all $t\in[0,T]$ and $\varphi\in C^\infty$, $\langle\mathcal{M}^N_t,\varphi\rangle=\frac{\sqrt{2}}{\sqrt{N}}\sum_{i=1}^N\int_0^t\nabla\varphi(X_i)\cdot\mathrm{d}B^i_s$, $\quad\mathbb{P}$-a.s..

Applying Itô's formula to the interacting particle system (1.1), we derive the following SPDE representation for the fluctuation measures $(\eta_t^N)_{t\in[0,T]}$ i.e., for every $\varphi \in C^\infty(\mathbb{T}^2)$,

$$\begin{aligned}\langle \varphi, \eta_t^N \rangle &= \langle \varphi, \eta_0^N \rangle + \int_0^t \langle \Delta\varphi, \eta_s^N \rangle \,\mathrm{d}s + \frac{1}{2}\int_0^t \Big\langle \sigma\cdot\nabla\Big(\sigma\cdot\nabla\varphi\Big), \eta_s^N \Big\rangle \,\mathrm{d}s \\ &\quad + \sqrt{N}\int_0^t \langle \nabla\varphi, \mu_N(s) K * \mu_N(s)\rangle \,\mathrm{d}s - \sqrt{N}\int_0^t t\,\langle \nabla\varphi, v_s K * v_s\rangle \,\mathrm{d}s \\ &\quad + \sqrt{2}\int_0^t \frac{1}{\sqrt{N}}\sum_{i=1}^N \nabla\varphi(X_s^i)\cdot \mathrm{d}B_i(s) + \int_0^t \big\langle \sigma\cdot\nabla\varphi, \eta_s^N \big\rangle \,\mathrm{d}W_s. \end{aligned} \tag{3.1}$$

For simplicity, we define the following interacting terms $\mathcal{K}_t^N : C^\infty(\mathbb{T}^2) \to \mathbb{R}$

$$\mathcal{K}_t^N(\varphi) := \sqrt{N}\langle \nabla\varphi, K * \mu_N(t)\mu_N(t)\rangle - \sqrt{N}\langle \nabla\varphi, v_t K * v_t\rangle. \tag{3.2}$$

To establish the tightness of the fluctuation measures $\{\eta^N\}_{N\in\mathbb{N}}$ and the additive noise term $\{\mathcal{M}^N\}_{N\in\mathbb{N}}$, we first obtain the uniform estimates for $\eta_t^N, \nabla\cdot[K*\mu_N(t)\mu_N(t) - v_t K * v_t], \mathcal{K}_t^N(\varphi)$ in Section 3.1 following the estimates in [WZZ23]. We also establish an additional estimate Lemma 3.5 in Section 3.2, by exploiting the structure of (3.1). The proof of tightness is more complicated than the case without environmental noise W, since we have to deal with the new transport noise term $\sigma\cdot\nabla\eta_t^N \mathrm{d}W_t$ and the uniform bound for the relative entropy $\sup_{t\in[0,T]} H(F_t^N|\bar{F}_t^N)(\omega)$ in Lemma 2.10 depends on $\omega\in\Omega$. To address these challenges, we use a classical localization argument.

3.1. **Estimates on the relative entropy.** In this section, we apply the relative entropy method to obtain uniform estimates essential for the subsequent proof of tightness.

The core idea behind the relative entropy method is to employ the Donsker-Varadhan variational formula, which gives Lemma 2.8, to decompose the target integral into two terms. One term is the relative entropy, which is bounded almost surely (i.e., Lemma 2.10), while the other is an exponential-type integral that can be controlled using estimates [JW18, Theorem 4] and [WZZ23, Lemma 2.3]. Compared to [WZZ23], in the environmental noise case, the components $\{\bar{X}_i, i\in\mathbb{N}\}$ of the limiting nonlinear SDE (1.7) are no longer independent but conditionally independent and identically distributed, i.e., for $i\neq j$

$$\mathcal{L}(\bar{X}_i(t)|\mathcal{F}_T^W)(\mathrm{d}x) = \mathcal{L}(\bar{X}_j(t)|\mathcal{F}_T^W)(\mathrm{d}x) = v_t(\mathrm{d}x), \quad \mathbb{P}-a.s.$$

in the sense of $\mathcal{P}(\mathbb{T}^2)$ and

$$\bar{X}_i(t) \perp \bar{X}_j(t)|\mathcal{F}_T^W.$$

At this time, two target measures in Lemma 2.8 are considered as $F^N(t)(\mathrm{d}x^N)$ and $\bar{F}_N(t)(\mathrm{d}x^N)$ defined in (1.8) and (1.9).

Lemma 3.1. *For each $\alpha > 1$, there exist constants C_α and C such that for all $N\in\mathbb{N}$,*

(1)

$$\begin{aligned}&\mathbb{E}\Big[\|\mu_N(t) - v_t\|_{H^{-\alpha}}^2 \mid \mathcal{F}_T^W\Big] \\ \leqslant &\frac{C_\alpha}{N}(H(F_N(t)|\bar{F}_N(t)) + 1), \quad \forall t\in[0,T], \quad \mathbb{P}-a.s.,\end{aligned}$$

(2)

$$\begin{aligned}&\mathbb{E}\Big[\|\nabla\cdot[K*\mu_N(t)\mu_N(t) - v_t K * v_t]\|_{H^{-\alpha}}^2 \mid \mathcal{F}_T^W\Big] \\ \leqslant &\frac{C_\alpha}{N}(H(F_N(t)|\bar{F}_N(t)) + 1), \quad \forall t\in[0,T], \quad \mathbb{P}-a.s.,\end{aligned}$$

(3)

$$\mathbb{E}\Big[|\langle \varphi K * (\mu_N(t) - v_t), \mu_N(t) - v_t\rangle| \mid \mathcal{F}_T^W\Big]$$

$$\leqslant \frac{C}{N}(H(F_N(t)|\bar{F}_N(t))+1), \quad \forall t \in [0,T], \quad \mathbb{P}-a.s.,$$

where the random measures $F^N(t)(\mathrm{d}x^N)$ and $\bar{F}^N(t)(\mathrm{d}x^N)$ on torus $\mathbb{T}^{2N}$ are defined by (1.8), (1.9), and v_t is the unique probabilistically strong solution to the mean field limit equation (1.5) in the sense of Definitions 2.2 and 2.3.

Proof. We now obtain estimates concerning the fluctuation measures, based on the results in [WZZ23]. For example, we express the conditional expectation $\mathbb{E}\Big[\|\mu_N(t)-v_t\|^2_{H^{-\alpha}} \mid \mathcal{F}^W_T\Big]$ through the conditional law of particles $X^N(t)$ with respect to the environmental noise $\mathcal{F}^W_T$,

$$\mathbb{E}\Big[\|\mu_N(t)-v_t\|^2_{H^{-\alpha}} \mid \mathcal{F}^W_T\Big] = \int_{\mathbb{T}^{dN}} \|\mu_N - v_t\|^2_{H^{-\alpha}} F^N(t)(\mathrm{d}x^N),$$

where $F^N(t)(\mathrm{d}x^N) = \mathcal{L}(X^N(t)|\mathcal{F}^W_T)(\mathrm{d}x^N)$ is a random measure on torus $\mathbb{T}^{2N}$. Through Donsker-Varadhan variational formula i.e. Lemma 2.8, we have

$$\begin{aligned}&\mathbb{E}\Big[\|\mu_N(t)-v_t\|^2_{H^{-\alpha}} \mid \mathcal{F}^W_T\Big]\\ \leqslant&\frac{1}{\kappa N}\left(H(F_N(t)|\bar{F}_N(t)) + \log\int_{\mathbb{T}^{2N}} \exp\left(\kappa N\|\mu_N - v_t\|^2_{H^{-\alpha}}\right)\bar{F}_N(t)(\mathrm{d}x^N)\right),\end{aligned}$$

where $\bar{F}_N(t)(\mathrm{d}x^N) = \mathcal{L}((\bar{X}^N(t))|\mathcal{F}^W_T)(\mathrm{d}x^N) = v_t^{\otimes N}(\mathrm{d}x^N)$ is a random measure on torus $\mathbb{T}^{2N}$. The rest of the proof follows by the same analysis in [WZZ23, Lemma 2.6-Lemma 2.9]. □

From Lemma 2.10, we know that it holds almost surely that for all

$$\frac{1}{2e^{C_0 T}}\big(H(F_N(t)|\bar{F}_N(t))+1\big)\exp\left\{-\int_0^t C_0\|v_s\|^2_{H^4}\mathrm{d}s\right\} \leqslant 1,$$

where C_0 is a positive deterministic constant depending on $\|v_0\|_{L^2(\mathbb{T}^2)}$ and $\inf_{x\in\mathbb{T}^2} v_0$. By choosing $m = 2e^{C_0T} + C_0 > 1$, which depends on $\|v_0\|_{L^2(\mathbb{T}^2)}, \inf_{x\in\mathbb{T}^2} v_0$ and T, we then obtain

$$\frac{1}{m}\exp\left\{-\int_0^t m\|v_s\|^2_{H^4}\mathrm{d}s\right\}\big(H(F_N(t)|\bar{F}_N(t))+1\big) \leqslant 1, \mathbb{P}-a.s..$$

We then deduce the following result by Lemma 3.1.

Corollary 3.2. *For each $t \in [0,T]$, define the weight term for $f \in L^2([0,T];H^4)$, $\mathcal{R}_t(f) = \frac{1}{m}\exp\Big\{-\int_0^t m\|f_s\|^2_{H^4}\mathrm{d}s\Big\}$, where the deterministic constant $m>1$ depends on $\|v_0\|_{L^2(\mathbb{T}^2)}, \inf_{x\in\mathbb{T}^2} v_0$ and T. Then, for each $\alpha > 1$, there exist constants C_α and C such that for all $N \in \mathbb{N}$,*

(1)

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\mathcal{R}_t(v)\|\mu_N(t)-v_t\|^2_{H^{-\alpha}}\Big] \leqslant \frac{C_\alpha}{N},$$

(2)

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\mathcal{R}_t(v)\|\nabla\cdot[K*\mu_N(t)\mu_N(t) - v_tK*v_t]\|^2_{H^{-\alpha}}\Big] \leqslant \frac{C_\alpha}{N},$$

(3)

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\mathcal{R}_t(v)\mid\langle\varphi K*(\mu_N(t)-v_t), \mu_N(t)-v_t\rangle\mid\Big] \leqslant \frac{C}{N}.$$

Remark 3.3. *Since the initial values are i.i.d. random variables, the classical central limit theorem allows us to infer that for each $\varphi \in C^\infty(\mathbb{T}^2)$, we have*

$$\langle \eta_0^N, \varphi \rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N \Big[\varphi(X_i(0)) - \langle \varphi, \mu \rangle \Big] \xrightarrow{N\to\infty} \mathcal{N}\Big(0, \langle \varphi^2, \mu \rangle - \langle \varphi, \mu \rangle^2 \Big),$$

in the sense of distribution, where $\mathcal{N}(0,a)$ denotes the centered Gaussian distribution on $\mathbb{R}$ with variance a. Using Lemma 3.1 which gives the tightness of laws of $\{\eta^N(0)\}$ in $H^{-\alpha}$, we then conclude that for every $\alpha > 1$, η_0^N converges in distribution to some η_0 in $H^{-\alpha}$.

3.2. **Tightness.** In this section, we will prove the tightness of laws of $\{\mathcal{L}(v, \eta^N, \mathcal{M}^N, W), N \in \mathbb{N}\}$ on the Polish space $\mathcal{H} = \mathcal{V} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{W}$, which is given by (2.4). We begin by introducing the following pathwise representation of the additive noise part in the decomposition of (3.1), given by

$$\frac{\sqrt{2}}{\sqrt{N}} \sum_{i=1}^N \int_0^t \nabla\varphi(X_i) \cdot \mathrm{d}B_s^i,$$

for each $\varphi \in C^\infty(\mathbb{T}^2)$, along with its corresponding estimate. The proof is provided in [WZZ23].

Lemma 3.4. *For each N, there exists a progressively measurable process $\mathcal{M}^N$ with values in $H^{-\alpha}$, for every $\alpha > 2$, such that*

(1) For all $t \in [0,T]$ and $\varphi \in C^\infty(\mathbb{T}^d)$, it holds $\mathbb{P}$-a.s.,

$$\langle \mathcal{M}_t^N, \varphi \rangle = \frac{\sqrt{2}}{\sqrt{N}} \sum_{i=1}^N \int_0^t \nabla\varphi(X_i) \cdot \mathrm{d}B_s^i,$$

(2) For every $\alpha > 2$, $\theta' \in (0, \frac{1}{2})$, there exists constants $C_{T,\alpha,\theta'}$ and $C_{T,\alpha}$ such that

$$\sup_N \mathbb{E}(\|\mathcal{M}^N\|^2_{C^{\theta'}([0,T],H^{-\alpha})}) \leqslant C_{T,\alpha,\theta'},$$
$$\sup_N \mathbb{E}\Big[\sup_{t\in[0,T]} \|\mathcal{M}_t^N\|^2_{H^{-\alpha}} \Big] \leqslant C_{T,\alpha}.$$

Furthermore, for every $\alpha > 2$, the sequence $(\mathcal{M}^N)_{N\in\mathbb{N}}$ is tight in the space $C([0,T], H^{-\alpha})$.

Before proceeding, we introduce the following stopping times $\{\tau_R, R > 0\}$.

$$\tau_R := \inf \left\{ 0 < t \leqslant T : \mathcal{R}_t^{-1}(v) = m \exp\left\{ \int_0^t m\|v_s\|^2_{H^4} \mathrm{d}s \right\} > R \right\},$$

(with the convention $\inf \emptyset = T$) and define

$$\eta_R^N(t) := \eta^N(t \wedge \tau_R), \quad t \in [0,T].$$

Then, we obtain the uniform estmates on η^N before the stopping time τ_R through Corollary 3.2.

Lemma 3.5. *For every $\alpha > 3$ and $R > 0$, there exists a constant $C_{\alpha,\sigma,T,R}$ such that*

$$\sup_N \mathbb{E} \sup_{t\in[0,T]} \|\eta_R^N(t)\|^2_{H^{-\alpha}} \leqslant C_{\alpha,\sigma,T,R}.$$

Proof. Notice that for each $N \in \mathbb{N}$, it holds $\mathbb{P}$-a.s. that for every $t \in [0,T]$,

$$\eta^N_{t\wedge\tau_R} - \eta_0^N = -\sqrt{N} \int_0^{t\wedge\tau_R} \nabla \cdot (\mu_N(s) K * \mu_N(s)) \mathrm{d}s + \sqrt{N} \int_0^{t\wedge\tau_R} \nabla \cdot (v_s K * v_s) \mathrm{d}s$$
$$+ \int_0^{t\wedge\tau_R} \Delta \eta_s^N \mathrm{d}s + \frac{1}{2} \int_0^{t\wedge\tau_R} \sigma \cdot \nabla\Big(\sigma \cdot \nabla \eta_s^N\Big) \mathrm{d}s + \mathcal{M}^N_{t\wedge\tau_R} - \int_0^{t\wedge\tau_R} \sigma \cdot \nabla \eta_s^N \mathrm{d}W_s.$$

Since $\mathcal{R}_t^{-1}(v) \leqslant R$ before the stopping time τ_R, we then have

$$\sup_{t\in[0,T]} \left\|\eta^N_{t\wedge\tau_R}\right\|^2_{H^{-\alpha}} \lesssim \|\eta^N_0\|^2_{H^{-\alpha}} + \sum_{i=1}^{5} J_i,$$

where

$$J_1 := \int_0^T R\mathcal{R}_s(v)\|\Delta\eta^N_s\|^2_{H^{-\alpha}}\mathrm{d}s,$$
$$J_2 := \int_0^T R\mathcal{R}_s(v)\|\sigma\cdot\nabla\Big(\sigma\cdot\nabla\eta^N_s\Big)\|^2_{H^{-\alpha}}\mathrm{d}s,$$
$$J_3 := \int_0^T R\mathcal{R}_s(v)N\|\nabla\Big(\mu_N(s)K*\mu_N(s) - v_sK*v_s\Big)\|^2_{H^{-\alpha}}\mathrm{d}s,$$
$$J_4 := \sup_{t\in[0,T]}\|\mathcal{M}^N_t\|^2_{H^{-\alpha}},$$
$$J_5 := \sup_{t\in[0,T]}\|\int_0^{t\wedge\tau_R}\sigma\cdot\nabla\eta^N_s\mathrm{d}W_s\|^2_{H^{-\alpha}}.$$

Observe that Lemma 2.13 implies $\|\sigma\cdot\nabla\Big(\sigma\cdot\nabla\eta^N_s\Big)\|^2_{H^{-\alpha}} \leqslant C_\sigma\|\eta^N_s\|^2_{H^{-\alpha+2}}$. By applying Corollary 3.2, we then have

$$\sup_N \mathbb{E}[J_2] \leqslant C_\sigma RT \sup_{t\in[0,T]}\mathbb{E}\Big[\mathcal{R}_t(v)\|\eta^N_t\|^2_{H^{-\alpha+2}}\Big] \leqslant RTC_{\alpha,\sigma}.$$

Similarly, by Corollary 3.2, we have

$$\sup_N \mathbb{E}[J_1+J_3] \leqslant RTC_{\alpha,\sigma}.$$

Recall that we have already established $\sup_N\mathbb{E}[J_4] \leqslant C_{T,\alpha}$ in Lemma 3.4. For J_5, applying Burkholder-Davis-Gundy's inequality, we have

$$\begin{aligned}\mathbb{E}[J_5] &\leqslant \mathbb{E}\int_0^{T\wedge\tau_R}\|\sigma\cdot\nabla\eta^N_s\|^2_{H^{-\alpha}}\mathrm{d}s\\ &\leqslant C_\sigma RT\sup_{t\in[0,T]}\mathbb{E}\Big[\mathcal{R}_t(v)\|\eta^N_t\|^2_{H^{-\alpha+1}}\Big] \leqslant C_{\alpha,\sigma,T,R}.\end{aligned}$$

Summerizing the estimates above, the proof is then completed. □

For $\alpha > 3$, we define the following stopping times $\{\tau_{\alpha,M,R}, M, R > 0\}$,

$$\tau^N_{\alpha,M,R} := \inf\left\{0 < t \leqslant T : \sup_{s\in[0,t]}\|\eta^N_s\|^2_{H^{-\alpha}} > M\right\}\wedge\tau_R,$$

(with the convention $\inf\emptyset = T$) and the associated stopped process

$$\eta^N_{\alpha,M,R}(t) := \eta^N(t\wedge\tau^N_{\alpha,M,R}),\quad t\in[0,T].$$

Based on Lemma 3.5, we derive a uniform estimate for the Hölder semi-norm of η^N.

Lemma 3.6. *For every $\alpha > 3$ and $\theta\in(0,\frac{1}{2})$, there exists a constant $C_{\alpha,\sigma,\theta,R,M,T}$ such that for every $R > 0$ and $M > 0$,*

$$\sup_N \mathbb{E}\Big[\|\eta^N_{\alpha,M,R}\|_{C^\theta([0,T],H^{-\alpha})}\Big] \leqslant C_{\alpha,\sigma,\theta,R,M,T}, \tag{3.3}$$

where

$$\|f_t\|_{C^\theta([0,T],H^{-\alpha})} := \sup_{0\leqslant s<t\leqslant T}\frac{\|f_t - f_s\|_{H^{-\alpha}}}{(t-s)^\theta}.$$

Proof. Notice that for each $N\in\mathbb{N}$, it holds $\mathbb{P}$-a.s. that for every $t\in[0,T]$,

$$\begin{aligned}\eta^N_{t\wedge\tau^N_{\alpha,M,R}}-\eta^N_0=&-\sqrt{N}\int_0^{t\wedge\tau^N_{\alpha,M,R}}\nabla\cdot(\mu_N(s)K*\mu_N(s))\mathrm{d}s+\sqrt{N}\int_0^{t\wedge\tau^N_{\alpha,M,R}}\nabla\cdot(v_sK*v_s)\mathrm{d}s\\&+\int_0^{t\wedge\tau^N_{\alpha,M,R}}\Delta\eta^N_s\mathrm{d}s+\frac{1}{2}\int_0^{t\wedge\tau^N_{\alpha,M,R}}\sigma\cdot\nabla\Big(\sigma\cdot\nabla\eta^N_s\Big)\mathrm{d}s\\&+\mathcal{M}^N_{t\wedge\tau^N_{\alpha,M,R}}-\int_0^{t\wedge\tau^N_{\alpha,M,R}}\sigma\cdot\nabla\eta^N_s\mathrm{d}W_s.\end{aligned}$$

Thus, $\{\|\eta^N_t-\eta^N_s\|_{H^{-\alpha}},\,0\leqslant s<t<T\}$ thus can be controlled by the following relation

$$\|\eta^N_{\alpha,M,R}(t)-\eta^N_{\alpha,M,R}(s)\|_{H^{-\alpha}}\lesssim\sum_{i=1}^{7}J^i_{s,t},\tag{3.4}$$

where $J^i_{s,t}$, $i=1,\dots,5$, are defined as

$$\begin{aligned}J^1_{s,t}&:=\left\|\int_s^t I_{[0,\tau^N_{\alpha,M,R}]}(r)\Delta\eta^N_r\mathrm{d}r\right\|_{H^{-\alpha}},\qquad J^2_{s,t}:=\left\|\int_s^t I_{[0,\tau^N_{\alpha,M,R}]}(r)\mathcal{K}^N_r\mathrm{d}r\right\|_{H^{-\alpha}},\\J^3_{s,t}&:=\left\|\int_s^t I_{[0,\tau^N_{\alpha,M,R}]}(r)\sigma\cdot\nabla(\sigma\cdot\nabla\eta^N_r)\mathrm{d}r\right\|_{H^{-\alpha}},\quad J^4_{s,t}:=\left\|\mathcal{M}^N_{t\wedge\tau^N_{\alpha,M,R}}-\mathcal{M}^N_{s\wedge\tau^N_{\alpha,M,R}}\right\|_{H^{-\alpha}},\\J^5_{s,t}&:=\left\|\int_s^t I_{[0,\tau^N_{\alpha,M,R}]}(r)\sigma\cdot\nabla\eta^N_r\mathrm{d}W_r\right\|_{H^{-\alpha}}.\end{aligned}$$

where $\mathcal{K}^N_t=\sqrt{N}\nabla\cdot[K*\mu_N(t)\mu_N(t)-K*v_tv_t]$.

For the drift terms $J^i_{s,t}, i=1,2,3$, it is sufficient to prove that

$$\sup_N\mathbb{E}\left(\sup_{0\leqslant s<t\leqslant T}\frac{J^i_{s,t}}{(t-s)^{\frac12}}\right)^2<C.$$

Indeed, for every $\theta\in(0,\frac12)$, we have the following estimate.

$$\mathbb{E}\left(\sup_{0\leqslant s<t\leqslant T}\frac{J^i_{s,t}}{(t-s)^{\theta}}\right)\leqslant\mathbb{E}\left(\sup_{0\leqslant s<t\leqslant T}\frac{J^i_{s,t}}{(t-s)^{\frac12}}\right)T^{\frac12-\theta}\leqslant\left[\mathbb{E}\left(\sup_{0\leqslant s<t\leqslant T}\frac{J^i_{s,t}}{(t-s)^{\frac12}}\right)^2\right]^{\frac12}T^{\frac12-\theta}.$$

For $J^1_{s,t}$, note that $\mathcal{R}^{-1}_t(v)=m\exp\left\{\int_0^t m\|v_s\|^2_{H^4}\mathrm{d}s\right\}\leqslant R$ before the stopping time $\tau_{\alpha,M,R}$, and we apply Hölder's inequality to derive the following estimate

$$\begin{aligned}\sup_N\mathbb{E}\left(\sup_{0\leqslant s<t\leqslant T}\frac{J^1_{s,t}}{(t-s)^{\frac12}}\right)^2&\leqslant\sup_N\mathbb{E}\Big[\int_0^T I_{[0,\tau^N_{\alpha,M,R}]}(t)\|\Delta\eta^N_t\|^2_{H^{-\alpha}}\mathrm{d}t\Big]\\&\leqslant\sup_N\mathbb{E}\Big[\int_0^T R\mathcal{R}_t(v)\|\Delta\eta^N_t\|^2_{H^{-\alpha}}\mathrm{d}t\Big]\\&\leqslant RT\sup_N\sup_{t\in[0,T]}\mathbb{E}\Big[\mathcal{R}_t(v)\|\Delta\eta^N_t\|^2_{H^{-\alpha}}\Big]\leqslant RTC_\alpha,\end{aligned}\tag{3.5}$$

where the final inequality follows from Corollary 3.2. Using a similar approach, we have

$$\sup_N\mathbb{E}\left(\sup_{0\leqslant s<t\leqslant T}\frac{J^2_{s,t}}{(t-s)^{\frac12}}\right)^2\leqslant RT\sup_N\sup_{t\in[0,T]}\mathbb{E}\Big[\mathcal{R}_t(v)\|\mathcal{K}^N_t\|^2_{H^{-\alpha}}\Big]\leqslant RTC_\alpha,\tag{3.6}$$

where we used Corollary 3.2 to reach the final inequality, and

$$\begin{aligned}\sup_N \mathbb{E}\left(\sup_{0\leqslant s<t\leqslant T}\frac{J^3_{s,t}}{(t-s)^{\frac{1}{2}}}\right)^2 \leqslant & RT\sup_N\sup_{t\in[0,T]}\mathbb{E}\Big[\mathcal{R}_t(v)\|\sigma\cdot\nabla(\sigma\cdot\nabla\eta^N_t)\|^2_{H^{-\alpha}}\Big] \\ \leqslant & C_\sigma RT\sup_N\sup_{t\in[0,T]}\mathbb{E}\Big[\mathcal{R}_t(v)\|\eta^N_t\|^2_{H^{-\alpha+2}}\Big]\leqslant RTC_{\alpha,\sigma}.\end{aligned} \tag{3.7}$$

where we applied Lemma 2.13 to get the second inequality and Lemma 3.2 to get the final inequality.

Followng the same approach in Lemma 3.4, i.e., [WZZ23, Lemma 3.2], we conclude that for any $\theta\in(0,\frac{1}{2})$,

$$\sup_N\mathbb{E}\Big(\sup_{0\leqslant s<t\leqslant T}\frac{J^4_{s,t}}{|t-s|^\theta}\Big)\leqslant C_{\alpha,\theta}. \tag{3.8}$$

For $J^5_{s,t}$, applying Burkholder-Davis-Gundy's inequality [BFH18, Theorem 2.3.8] gives, for any $\theta'>1$,

$$\begin{aligned}\sup_N\mathbb{E}(|J^5_{s,t}|^{2\theta'})\leqslant & C_{\theta'}\sup_N\mathbb{E}\Big[\int_s^t I_{[0,\tau^N_{\alpha,M,R}]}(r)\|\sigma\cdot\nabla\eta^N_R\|^2_{H^{-\alpha}}\mathrm{d}r\Big]^{\theta'} \\ \leqslant & C_{\theta',\sigma}\sup_N\mathbb{E}\Big[\int_s^t I_{[0,\tau^N_{\alpha,M,R}]}(r)\sup_{s\in[0,r]}\|\eta^N_s\|^2_{H^{-\alpha}}\mathrm{d}r\Big]^{\theta'}\leqslant C_{\theta',\sigma,M}\mid t-s\mid^{\theta'},\end{aligned}$$

where in the final inequality we use the fact that for $\mathbb{P}$-a.s., $\sup_{s\in[0,t]}\|\eta^N_R(s)\|^2_{H^{-\alpha}}\leqslant M$ up to the stopping time $\tau_{\alpha,M,R}$. By applying the Kolmogorov continuity theorem [BFH18, Theorem 2.3.11], we then deduce that for any $\theta\in(0,\frac{1}{2})$,

$$\sup_N\mathbb{E}\Big(\sup_{0\leqslant s<t\leqslant T}\frac{J^5_{s,t}}{|t-s|^\theta}\Big)\leqslant C_{\theta,\sigma,M}. \tag{3.9}$$

The result follows by combining inequalities (3.5)-(3.9). □

Then, we demonstrate the tightness of the fluctuation measures $(\eta^N)_{N\geqslant1}$ in the Polish space

$$\mathcal{X}=\left\{\bigcap_{k\in\mathbb{N}}\Big[C([0,T],H^{-3-\frac{1}{k}})\cap L^2([0,T],H^{-1-\frac{1}{k}})\Big]\right\}.$$

Lemma 3.7. *The sequence of laws of $(\eta^N)_{N\geqslant1}$ is tight in the space $\mathcal{X}$.*

Proof. First, we claim that it is sufficient to prove the tightness of the sequence of laws of $(\eta^N)_{N\geqslant1}$ in the space

$$\mathcal{X}_k:=C([0,T],H^{-3-\frac{1}{k}})\cap L^2([0,T],H^{-1-\frac{1}{k}})$$

for each fixed $k\in\mathbb{N}$. Indeed, if the sequence of laws of $(\eta^N)_{N\geqslant1}$ is tight in the space $\mathcal{X}_k$ for each k, then for any $\vartheta>0$, we can choose compact sets A^ϑ_k in $C([0,T],H^{-3-\frac{1}{k}})\cap L^2([0,T],H^{-1-\frac{1}{k}})$ for each $k\in\mathbb{N}$ such that

$$\mathbb{P}(\eta^N\notin A^\vartheta_k)<\vartheta 2^{-k},\quad\forall N\in\mathbb{N}.$$

The set A^ϑ in $\mathcal{X}$ defined by

$$A^\vartheta:=\bigcap_{k\in\mathbb{N}}A^\vartheta_k$$

is compact and satisfies

$$\mathbb{P}\Big(\eta^N\notin A^\vartheta\Big)\leqslant\sum_{k\in\mathbb{N}}\mathbb{P}(\eta^N\notin A^\vartheta_k)<\vartheta,\quad\forall N\in\mathbb{N},$$

which implies $(\eta^N)_{N\geqslant1}$ is tight in the space $\mathcal{X}$.

Next, we prove that the sequence $(\eta^N)_{N\geqslant 1}$ is tight in the space $C([0,T],H^{-\alpha-2})\cap L^2([0,T],H^{-\alpha})$, for every $\alpha>1$. For any $\delta>0$ and $\alpha>\alpha'>1$, we define

$$K^\delta := \Big\{\eta \Big| \sup_{t\in[0,T]}\|\eta_t\|_{H^{-\alpha'-2}}\leqslant \frac{1}{\delta}, \int_0^T \|\eta_t\|^2_{H^{-\frac{2\alpha+2}{4}}}\mathrm{d}t \leqslant \frac{1}{\delta}, \|\eta\|_{C^{\frac{1}{8}}([0,T],H^{-\alpha-2})}\leqslant \frac{1}{\delta}\Big\}$$

which is a compact subset of $C([0,T],H^{-\alpha-2})\cap L^2([0,T],H^{-\alpha})$ as established by [BFH18, corollary 1.8.4] and Arzela-Ascoli theorem [Kel17, Theorem 7.17]. Recall the stopping times and stopped processes,

$$\tau_R = \inf\Big\{0<t\leqslant T : \mathcal{R}_t^{-1}(v) = m\exp\Big\{\int_0^t m\|v_s\|^2_{H^4}\mathrm{d}s\Big\} > R\Big\},$$
$$\tau^N_{\alpha+2,M,R} = \inf\Big\{0<t\leqslant T : \sup_{s\in[0,t]}\|\eta^N_R(s)\|^2_{H^{-\alpha-2}} > M\Big\}\wedge\tau_R,$$

and

$$\eta^N_R(t) = \eta^N(t\wedge\tau_R), \quad t\in[0,T],$$
$$\eta^N_{\alpha+2,M,R}(t) = \eta^N(t\wedge\tau^N_{\alpha+2,M,R}), \quad t\in[0,T],$$

we can directly conclude that for every $R>0$ and $M>0$,

$$\begin{aligned}
&\mathbb{P}\Big(\eta^N\notin K^\delta\Big)\\
\leqslant&\mathbb{P}\Big(\eta^N=\eta^N_R=\eta^N_{\alpha+2,M,R}\notin K^\delta, m\exp\Big\{\int_0^T m\|v_s\|^2_{H^4}\mathrm{d}s\Big\}\leqslant R, \sup_{t\in[0,T]}\|\eta^N_R(t)\|^2_{H^{-\alpha-2}}\leqslant M\Big)\\
+&\mathbb{P}\Big(m\exp\Big\{\int_0^T m\|v_s\|^2_{H^4}\mathrm{d}s\Big\}\geqslant R\Big)+\mathbb{P}\Big(\sup_{t\in[0,T]}\|\eta^N_R(t)\|^2_{H^{-\alpha-2}}\geqslant M\Big).
\end{aligned}$$

Given $\varepsilon>0$, since $\mathcal{R}_T^{-1}(v) = m\exp\Big\{\int_0^T m\|v_s\|^2_{H^4}\mathrm{d}s\Big\}<\infty, \mathbb{P}$-a.s., we can thus choose a sufficiently large $R_0>0$ such that

$$\mathbb{P}\Big(m\exp\Big\{\int_0^T m\|v_s\|^2_{H^4}\mathrm{d}s\Big\}\geqslant R_0\Big)\leqslant\frac{\varepsilon}{4}.$$

By Chebyshev's inequality, we have

$$\mathbb{P}\Big(\sup_{t\in[0,T]}\|\eta^N_{R_0}(t)\|^2_{H^{-\alpha-2}}\geqslant M\Big)\leqslant\frac{\mathbb{E}\Big[\sup_{t\in[0,T]}\|\eta^N_{R_0}(t)\|^2_{H^{-\alpha-2}}\Big]}{M}.$$

Using Lemma 3.5, we can choose a sufficiently large $M_0>0$ such that for all $N\in\mathbb{N}$,

$$\mathbb{P}\Big(\sup_{t\in[0,T]}\|\eta^N_{R_0}(t)\|^2_{H^{-\alpha-2}}\geqslant M_0\Big)\leqslant\frac{\varepsilon}{4}.$$

We then conclude that

$$\begin{aligned}
\mathbb{P}\Big(\eta^N\notin K^\delta\Big)\leqslant\ &\mathbb{P}\Big(\eta^N=\eta^N_{R_0}=\eta^N_{\alpha+2,M_0,R_0}\notin K^\delta, m\exp\Big\{\int_0^T m\|v_s\|^2_{H^4}\mathrm{d}s\Big\}\leqslant R_0,\\
&\text{and }\sup_{t\in[0,T]}\|\eta^N_{R_0}(t)\|^2_{H^{-\alpha-2}}\leqslant M_0\Big)+\frac{\varepsilon}{2}.\\
\leqslant\ &\mathbb{P}\Big(\sup_{t\in[0,T]}\|\eta^N_{R_0}(t)\|_{H^{-\alpha'-2}}\geqslant\frac{1}{\delta}\Big)+\mathbb{P}\Big(\|\eta^N_{\alpha+2,M_0,R_0}\|_{C^{\frac{1}{8}}([0,T],H^{-\alpha-2})}\geqslant\frac{1}{\delta}\Big)\\
+\ &\mathbb{P}\Big(R_0\int_0^T\mathcal{R}_t(v)\|\eta^N_t\|^2_{H^{-\frac{2\alpha+2}{4}}}\mathrm{d}t\geqslant\frac{1}{\delta}\Big)+\frac{\varepsilon}{2}.
\end{aligned}$$

Applying Chebyshev's inequality again, along with Corollary 3.2, Lemma 3.5 and Lemma 3.6, we can select a sufficiently small $\delta_0 > 0$ such that $\mathbb{P}\Big(\eta^N \notin K^{\delta_0}\Big) < \varepsilon$, which completes the proof. □

It is well known that every probability measure on a Polish space is tight. Then, we have the tightness of laws of v and W. Combining Lemma 3.4 and Lemma 3.7, we conclude the following result.

Lemma 3.8. *The law of* $\Big(v, \eta^N, \mathcal{M}^N, W\Big)$ *with values in* $\mathcal{H}$ *as defined in* (2.4) *is tight.*

By Skorohod's representation, we obtain the following result.

Proposition 3.9. *There exists a subsequence of* $(v, \eta^N, \mathcal{M}^N, W)_{N\geqslant 1}$*, still denoted by* $(v, \eta^N, \mathcal{M}^N, W)$ *for simplicity, and a probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *with* $\mathcal{H}$*-valued random variables* $(\tilde{v}^N, \tilde{\eta}^N, \tilde{\mathcal{M}}^N, \tilde{W}^N)_{N\geqslant 1}$ *and* $(\tilde{v}, \tilde{\eta}, \tilde{M}, \tilde{W})$ *such that*

1. *For each* $N \in \mathbb{N}$*, the law of* $(\tilde{v}^N, \tilde{\eta}^N, \tilde{\mathcal{M}}^N, \tilde{W}^N)$ *coincides with the law of* $(v, \eta^N, \mathcal{M}^N, W)$*.*
2. *The sequence of* $\mathcal{H}$*-valued random variables* $(\tilde{v}^N, \tilde{\eta}^N, \tilde{\mathcal{M}}^N, \tilde{W}^N)_{N\geqslant 1}$ *converges to* $(\tilde{v}, \tilde{\eta}, \tilde{M}, \tilde{W})$ *in* $\mathcal{H}$ $\tilde{\mathbb{P}}$*-a.s..*

Remark 3.10. *We emphasize that* $(\tilde{v}^N, \tilde{W}^N)_{N\geqslant 1}$ *are different random variables, but they share the same law on the new probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$*.*

4. Well posedness of the fluctuation SPDE

This section aims to establish the well-posedness of the fluctuation SPDE (1.11) and the convergence from the fluctuation measures $(\tilde{\eta}^N)_{N\geqslant 1}$ to the probabilistically strong solution $\tilde{\eta}$ of the fluctuation SPDE (1.11).

4.1. Identification of the limiting points.

In this subsection, we identify the limiting points of the fluctuation process $(\tilde{\eta}^N)_{N\geqslant 1}$ as probabilistically weak solutions to the fluctuation SPDE (1.11). The proof proceeds in several steps. First, we identify the joint law of $(\tilde{M}, \tilde{W})$ in Lemma 4.2. Next, we study the convergence of the interacting term $\tilde{\mathcal{K}}_t^N(\varphi)$ in Lemma 4.4. In the third step, we show that the limit process $\tilde{v}$, given in Propositon 3.9, is the unique probabilistically strong solution to the mean field limit equation (1.5), as shown in Proposition 4.6. Finally, we prove the convergence of the transport noise term $\sigma \cdot \nabla\tilde{\eta}_t \mathrm{d}\tilde{W}_t$ in Proposition 4.7.

There are two challenges in this section. Firstly, unlike in the case without environmental noise W, we must address not only the convergence of the fluctuation measures $(\tilde{\eta}^N)_{N\geqslant 1}$ and the additive noise $(\tilde{\mathcal{M}}^N)_{N\geqslant 1}$, but also the convergence of the new multiplicative noise term $\sigma \cdot \nabla\tilde{\eta}^N \mathrm{d}\tilde{W}_t^N$, as well as the convergence of the random mean field limit $\tilde{v}^N$. Furthermore, we identifying the joint law of $(\tilde{M}, \tilde{W})$ by studying the conditional law of additive noise $\{\tilde{\mathcal{M}}_t, t \in [0, T]\}$ with respect to the environmental noise $\mathcal{F}_T^W$, which is the main challenge in this section. To address this, we establish the following strong convergence from the interacting particle system (1.1) to the conditional Mckean-Vlasov equation (1.7).

Proposition 4.1. *For* $i \in \mathbb{N}$ *and* $t \in [0, T]$*, we have* [2]

$$\lim_{N\to\infty} \mathbb{E}\big|X_i^N(t) - \bar{X}_i(t)\big|^2 = 0, \tag{4.1}$$

where X^N *is the unique probabilistically strong solution to the stochastic point vortex model* (1.1) *with i.i.d. initial values* $X^N(0) = (X_1(0), \cdots, X_N(0))$ *and* $\bar{X}_i$ *is the unique probabilistically strong solution to the conditional Mckean-Vlasov equation* (1.7) *with initial value* $X_i(0)$*.*

[2] The well-posedness of stochastic point vortex model (1.1) and conditional Mckean-Vlasov equation (1.7) has been given in [SZ24, Lemma 3.4, Proposition 4.1].

Proof. Consider the difference between the particle system (1.1) and the conditional Mckean-Vlasov equation (1.7),

$$
\begin{aligned}
X_i^N(t)-\bar{X}_i(t) &= \int_0^t \Big(K*\mu_N(s)(X_i^N(s)) - K*v_s(\bar{X}_i(s))\Big)\mathrm{d}s \\
&\quad + \int_0^t \Big(\sigma(X_i^N(s)) - \sigma(\bar{X}_i(s))\Big)\circ \mathrm{d}W_s \\
&= \int_0^t \Big(K*\mu_N(s)(X_i^N(s)) - K*v_s(\bar{X}_i(s))\Big)\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \Big((\sigma\cdot\nabla\sigma)(X_i^N(s)) - (\sigma\cdot\nabla\sigma)(\bar{X}_i(s))\Big)\mathrm{d}s + \int_0^t \Big(\sigma(X_i^N(s)) - \sigma(\bar{X}_i(s))\Big)\mathrm{d}W_s.
\end{aligned}
$$

where we convert the Stratonovich integral into Itô's integral in the last equality.

In our previous work [SZ24, Lemma 3.2], we showed that $v\in C([0,T];H^2)$, $\mathbb{P}$-a.s.. We now define the stopping times $\{\Theta_R, R>0\}$ (with the convention $\inf\emptyset = T$) as follows.

$$
\Theta_R := \inf\Big\{0<t\leqslant T \big| H_t(v) := \sup_{s\in[0,t]} \|v_s\|_{H^2} + m\exp\Big\{\int_0^t m\|v_s\|_{H^4}^2 \mathrm{d}s\Big\} > R\Big\}.
$$

Applying Itô's formula, we obtain

$$
\begin{aligned}
\mid X_i^N(t\wedge\Theta_R) - \bar{X}_i(t\wedge\Theta_R)\mid^2 &= \int_0^{t\wedge\Theta_R} (X_i^N(s)-\bar{X}_i(s))\Big(K*\mu_N(s)(X_i^N(s)) - K*v_s(\bar{X}_i(s))\Big)\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^{t\wedge\Theta_R} (X_i^N(s)-\bar{X}_i(s))\Big((\sigma\cdot\nabla\sigma)(X_i^N(s)) - (\sigma\cdot\nabla\sigma)(\bar{X}_i(s))\Big)\mathrm{d}s \\
&\quad + \int_0^{t\wedge\Theta_R} (X_i^N(s)-\bar{X}_i(s))\Big(\sigma(X_i^N(s)) - \sigma(\bar{X}_i(s))\Big)\mathrm{d}W_s \\
&\quad + \int_0^{t\wedge\Theta_R} \Big(\sigma(X_i^N(s)) - \sigma(\bar{X}_i(s))\Big)^2\mathrm{d}s.
\end{aligned}
$$

Direct computation yields the following identity

$$
\begin{aligned}
\mid X_i^N(t\wedge\Theta_R) - \bar{X}_i(t\wedge\Theta_R)\mid^2 &= \int_0^{t\wedge\Theta_R} (X_i^N(s)-\bar{X}_i(s))\Big(K*\mu_N(s)(X_i^N(s)) - K*v_s(X_i^N(s))\Big)\mathrm{d}s \\
&\quad + \int_0^{t\wedge\Theta_R} (X_i^N(s)-\bar{X}_i(s))\Big(K*v_s(X_i^N(s)) - K*v_s(\bar{X}_i(s))\Big)\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^{t\wedge\Theta_R} (X_i^N(s)-\bar{X}_i(s))\Big((\sigma\cdot\nabla\sigma)(X_i^N(s)) - (\sigma\cdot\nabla\sigma)(\bar{X}_i(s))\Big)\mathrm{d}s \\
&\quad + \int_0^{t\wedge\Theta_R} (X_i^N(s)-\bar{X}_i(s))\Big(\sigma(X_i^N(s)) - \sigma(\bar{X}_i(s))\Big)\mathrm{d}W_s \\
&\quad + \int_0^{t\wedge\Theta_R} \Big(\sigma(X_i^N(s)) - \sigma(\bar{X}_i(s))\Big)^2\mathrm{d}s.
\end{aligned}
$$

By Sobolev embedding theorem, we deduce that

$$
\|K*v_s\|_{C^1} \leqslant \|K*v_s\|_{H^3} = \|\nabla^{\perp}(-\Delta)^{-1}v_s\|_{H^3} \leqslant \|v_s\|_{H^2}. \tag{4.2}
$$

Here we used the fact the Biot-Savart kernel $K=\nabla^{\perp}G$, where G is the Green function of $-\Delta$ on torus, as given in [FGP11]. Through the compactness of torus, together with the Lipschitz property of $\sigma, \sigma\cdot\nabla\sigma, K*v_s$ and the estimate for $K*v_s$ given in (4.2), we obtain the following estimate.

$$
\mid X_i^N(t\wedge\Theta_R) - \bar{X}_i(t\wedge\Theta_R)\mid^2 \leqslant C\int_0^{t\wedge\Theta_R} \big|K*\mu_N(s)(X_i^N(s)) - K*v_s(X_i^N(s))\big|\mathrm{d}s
$$

$$+\int_0^t (C_\sigma + R) \mid X_i^N(s\wedge\Theta_R) - \bar{X}_i(s\wedge\Theta_R)\mid^2 \mathrm{d}s$$
$$+\int_0^{t\wedge\Theta_R} (X_i^N(s) - \bar{X}_i(s))\Big(\sigma(X_i^N(s)) - \sigma(\bar{X}_i(s))\Big)\mathrm{d}W_s.$$

Taking expectation of both sides, we then conclude that

$$\begin{aligned}\mathbb{E}\Big[\mid X_i^N(t\wedge\Theta_R) - \bar{X}_i(t\wedge\Theta_R)\mid^2\Big] \leqslant &(C_\sigma + R)\int_0^t \mathbb{E}\Big[\mid X_i^N(s\wedge\Theta_R) - \bar{X}_i(s\wedge\Theta_R)\mid^2\Big]\mathrm{d}s\\ &+ C\int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}} \mid K*\mu_N(s)(X_i^N(s)) - K*v_s(X_i^N(s))\mid\Big]\mathrm{d}s.\end{aligned}$$

We now deal with the interacting term $K*\mu_N(s)(X_i^N(s)) - K*v_s(X_i^N(s))$. As done in [FGP11, Section 3.2], we first regularize K by introducing smooth periodic functions K_ε such that $K_\varepsilon(x) = K(x)$ for any $|x| > \varepsilon$ and they satisfy

$$|K(x)| + |K_\varepsilon(x)| \lesssim \frac{1}{|x|}, \quad \forall\varepsilon > 0. \tag{4.3}$$

We decompose the interaction term into three parts.

$$K*\mu_N(s)(X_i^N(s)) - K*v_s(X_i^N(s)) = \sum_{i=1}^3 J_i,$$

where

$$\begin{aligned}J_1(s) &:= K*\mu_N(s)(X_i^N(s)) - K_\varepsilon*\mu_N(s)(X_i^N(s)),\\ J_2(s) &:= K_\varepsilon*\mu_N(s)(X_i^N(s)) - K_\varepsilon*v_s(X_i^N(s)),\\ J_3(s) &:= K_\varepsilon*v_s(X_i^N(s)) - K*v_s(X_i^N(s)).\end{aligned}$$

Recall that v is a continuous L^2-valued $\mathcal{F}_t^W$-adapted process, we then have

$$\begin{aligned}\int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}J_1(s)\Big]\mathrm{d}s =& \int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}\mathbb{E}\Big[\frac{1}{N}\sum_{j\neq i}(K-K_\varepsilon)(X_i^N(s) - X_j^N(s))|\mathcal{F}_T^W\Big]\Big]\mathrm{d}s\\ \leqslant&\frac{N-1}{N}\int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}\mathbb{E}\Big[|K-K_\varepsilon|(X_1^N(s) - X_2^N(s))|\mathcal{F}_T^W\Big]\Big]\mathrm{d}s,\end{aligned}$$

where the second inequality follows from the symmetry of the random measure $\mathcal{L}(X^N(t)|\mathcal{F}_T^W)(\mathrm{d}x^N)$ on $\mathbb{T}^{2N}$. The upper bound (4.3) for the Biot-Savart kernel K and its regularized version K_ε yields

$$\begin{aligned}\int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}J_1(s)\Big]\mathrm{d}s \lesssim& \int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}\int_{\mathbb{T}^2}\frac{1}{|x_1-x_2|}I_{\{|x_1-x_2|\leqslant\varepsilon\}}F^{N;2}(s)(\mathrm{d}x_1,\mathrm{d}x_2)\Big]\mathrm{d}s\\ \leqslant&\mathbb{E}\Big[\varepsilon^{\frac12}\int_{\mathbb{T}^2}I_{\{|x_1-x_2|\leqslant\varepsilon\}}\frac{1}{|x_1-x_2|^{\frac32}}F^{N;2}(s)(\mathrm{d}x_1,\mathrm{d}x_2)\Big]\mathrm{d}s,\end{aligned}$$

where $F^{N;2}(t)(\mathrm{d}x_1,\mathrm{d}x_2)$ is the 2-marginal of the random measure $F^N(t)(\mathrm{d}x^N) = \mathcal{L}(X^N(t)|\mathcal{F}_T^W)(\mathrm{d}x^N)$ on $\mathbb{T}^{2N}$. Using the property of Fisher information, i.e., Lemma 2.11, we conclude that

$$\begin{aligned}\int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}J_1(s)\Big]\mathrm{d}s \lesssim&\varepsilon^{\frac12}\mathbb{E}\int_0^T\Big[C(I(F^{N;2}(s))+1)\Big]\mathrm{d}s\\ \leqslant&\varepsilon^{\frac12}\mathbb{E}\int_0^T\Big[C(\frac{2}{N}I(F^N(s))+1)\Big]\mathrm{d}s \leqslant C_{v_0}\varepsilon^{\frac12}.\end{aligned}$$

Here we used the sub-additivity of Fisher information, i.e., $I(F^{N,k}(t)) \leqslant \frac{k}{N}I(F^N(t)), \forall k\leqslant N$, to derive the second inequality. For the final inequality, we applied the boundness about the Fisher information $I(F^N(t))$ for $F^N(t)(\mathrm{d}x^N)$, i.e., $\int_0^T I(F^N(t))\mathrm{d}t \leqslant C_{v_0}$, given in [SZ24, Lemma 4.2].

Moving on to J_2, by the Sobolev embedding theorem, we find that for $\alpha > 1$,

$$\begin{aligned}\int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}J_2(s)\Big]\mathrm{d}s &= \int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}\mathbb{E}\Big[K_\varepsilon * \mu_N(s)(X_i^N(s)) - K_\varepsilon * v_s(X_i^N(s))|\mathcal{F}_T^W\Big]\Big]\mathrm{d}s \\ &\leqslant \int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}\mathbb{E}\Big[\|K_\varepsilon * \mu_N(s) - K_\varepsilon * v_s\|_{L^\infty}|\mathcal{F}_T^W\Big]\mathrm{d}s \\ &\lesssim \int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}\mathbb{E}\Big[\|K_\varepsilon * \mu_N(s) - K_\varepsilon * v_s\|_{H^2}|\mathcal{F}_T^W\Big]\mathrm{d}s.\end{aligned}$$

Applying Lemma 2.15 yields

$$\begin{aligned}\int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}J_2(s)\Big]\mathrm{d}s &\lesssim \int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}\mathbb{E}\Big[\|K_\varepsilon\|_{B^{2+\alpha}_{1,2}}\|\mu_N(s) - v_s\|_{H^{-\alpha}}|\mathcal{F}_T^W\Big]\mathrm{d}s \\ &\leqslant \|K_\varepsilon\|_{B^{2+\alpha}_{1,2}}\int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}R\mathcal{R}_s(v)\|\mu_N(s) - v_s\|_{H^{-\alpha}}\Big]\mathrm{d}s \\ &\leqslant \|K_\varepsilon\|_{B^{2+\alpha}_{1,2}}T\frac{C_{R,\alpha}}{N},\end{aligned}$$

where the last inequality is derived using Corollary 3.2.

Finally, using the upper bound (4.3) of Biot-Savart kernel K and regularized version Biot-Savart kernel K_ε again, we have

$$\begin{aligned}\int_0^T \mathbb{E}\Big[I_{\{s\leqslant\Theta_R\}}J_3(s)\Big]\mathrm{d}s &\leqslant \int_0^T \mathbb{E}\Big[|K_\varepsilon * v_s(X_i^N(s)) - K * v_s(X_i^N(s))|\Big]\mathrm{d}s \\ &\lesssim \int_0^T \mathbb{E}\Big[\int_{\mathbb{T}^2}\frac{1}{|X_i^N(s)-y|}I_{\{|X_i^N-y|\leqslant\varepsilon\}}v_s(y)\mathrm{d}y\Big]\mathrm{d}s \\ &\leqslant C_{v_0}\varepsilon^{\frac{1}{2}}\int_0^T \mathbb{E}\Big[\int_{\mathbb{T}^2}\frac{1}{|X_i^N(s)-y|^{\frac{3}{2}}}\mathrm{d}y\Big]\mathrm{d}s \leqslant C_{v_0,T}\varepsilon^{\frac{1}{2}}.\end{aligned}$$

Here the third inequality follows from the regularity of v in Definition (2.1), i.e., $\|v_t\|_{L^\infty} \leqslant \|v_0\|_{L^\infty}, \forall t \in [0,T], \mathbb{P}$-a.s.. In summary, we conclude that

$$\begin{aligned}\mathbb{E}\Big[\mid X_i^N(t\wedge\Theta_R) - \bar{X}_i(s\wedge\Theta_R)\mid^2\Big] \leqslant&(C_\sigma + R)\int_0^t \mathbb{E}\Big[\mid X_i^N(s\wedge\Theta_R) - \bar{X}_i(s\wedge\Theta_R)\mid^2\Big]\mathrm{d}s \\ &+ C_{v_0,T}\varepsilon^{\frac{1}{2}} + \|K_\varepsilon\|_{B^{2+\alpha}_{1,2}}T\frac{C_{R,\alpha}}{N}.\end{aligned}$$

Choosing some $\alpha_0 > 1$ and applying Gronwall's lemma, we get

$$\mathbb{E}\Big[\mid X_i^N(t\wedge\Theta_R) - \bar{X}_i(t\wedge\Theta_R)\mid^2\Big] \leqslant \Big(C_{v_0,T}\varepsilon^{\frac{1}{2}} + \|K_\varepsilon\|_{B^{2+\alpha_0}_{1,2}}T\frac{C_{R,\alpha_0}}{N}\Big)\exp\{(C_\sigma + R)T\}.$$

Let $N \to \infty$ and then $\varepsilon \to 0$, we have $\lim\limits_{N\to\infty}\mathbb{E}\Big[\sup_{s\in[0,T]}\mid X_i^N(s\wedge\Theta_R) - \bar{X}_i(s\wedge\Theta_R)\mid^2\Big] = 0$, for every $R > 0$. Since the compactness of $\mathbb{T}^2$ ensures that $\sup_N \mathbb{E}[\mid X_i^N(t) - \bar{X}_i(t)\mid^2] < \infty$, we arrive at the desired result.

$$\begin{aligned}&\lim_{N\to\infty}\mathbb{E}[\mid X_i^N(t) - \bar{X}_i(t)\mid^2] \\ =&\lim_{N\to\infty}\sum_{n=0}^\infty \mathbb{E}\Big[\mid X_i^N(t\wedge\Theta_{n+1}) - \bar{X}_i(t\wedge\Theta_{n+1})\mid^2 I\{n\leqslant H_T(v) < n+1\}\Big] \\ =&\sum_{n=0}^\infty\lim_{N\to\infty}\mathbb{E}\Big[\mid X_i^N(t\wedge\Theta_{n+1}) - \bar{X}_i(t\wedge\Theta_{n+1})\mid^2 I\{n\leqslant H_T(v) < n+1\}\Big] \\ \leqslant&\sum_{n=0}^\infty\lim_{N\to\infty}\mathbb{E}\Big[\mid X_i^N(t\wedge\Theta_{n+1}) - \bar{X}_i(t\wedge\Theta_{n+1})\mid^2\Big] = 0.\end{aligned}$$

□

Now, let $(\mathcal{G}_t)_{t\in[0,T]}$ be the natural filtration of the process $(\tilde{v},\tilde{\eta},\tilde{\mathcal{M}},\tilde{W})$. That means that, for each $t\in[0,T]$, $\mathcal{G}_t$ is the smallest σ-algebra such that $\tilde{v}(s):\tilde{\Omega}\to L^2$, $\tilde{\eta}(s):\tilde{\Omega}\to H^{\alpha},\alpha=-3-\frac{1}{k},k\in\mathbb{N}$, $\tilde{\mathcal{M}}(s):\tilde{\Omega}\to H^{\gamma},\gamma=-2-\frac{1}{k},k\in\mathbb{N}$, and $\tilde{W}(s):\tilde{\Omega}\to\mathbb{R}$ are measurable for all $s\in[0,t]$. Let $\mathcal{N}:=\{M\in\tilde{\mathcal{F}}\mid\tilde{\mathbb{P}}(M)=0\}$. We will consider the augmented filtration $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$ which is defined by

$$\tilde{\mathcal{F}}_t:=\bigcap_{s>t}\sigma\left(\mathcal{G}_s\cup\mathcal{N}\right),\ t\in[0,T].$$

The augmented filtration $(\tilde{\mathcal{F}}_t)_t$ is a normal filtration. For $N\in\mathbb{N}$, we do the same construction to define the natural filtration $(\mathcal{G}^N_t)_{t\in[0,T]}$ and the corresponding augmented filtration $(\tilde{\mathcal{F}}^N_t)_{t\in[0,T]}$ of $(\tilde{v}^N,\tilde{\eta}^N,\tilde{\mathcal{M}}^N,\tilde{W}^N)$.

We now focus on the conditional law of $\{\tilde{\mathcal{M}}_t,t\in[0,T]\}$ with respect to the environmental noise $\mathcal{F}^{\tilde{W}}_T$. The basic idea, inspired by [KX04], is to consider a similar form of "$\{\mathcal{M}^N_t,t\in[0,T]\}$" derived from the conditionally i.i.d. particles $\{\bar{X}_i\}_{i\geqslant 1}$ of (1.7), instead of the interacting particle system (1.1). This requires the strong convergence result of X^N_i to $\bar{X}_i$, established in Proposition 4.1. Using the classical central limit theorem, we ultimately obtain that the conditional law of $\{\tilde{\mathcal{M}}_t,t\in[0,T]\}$ given by (4.4).

Lemma 4.2. *For every $\varphi\in C^{\infty}(\mathbb{T}^2)$ and $0\leqslant t<r+t\leqslant T$, it holds $\tilde{\mathbb{P}}$-a.s. that*

$$\begin{aligned}\tilde{\mathbb{E}}\Big[\exp\{i\langle\tilde{\mathcal{M}}_t,\varphi\rangle\}\mid\mathcal{F}^{\tilde{W}}_T\Big]&=\exp\Big\{-\int_0^t\langle|\nabla\varphi|^2,\tilde{v}_s\rangle\mathrm{d}s\Big\},\\ \mathbb{E}\Big[\exp i\Big\langle\varphi,(\tilde{\mathcal{M}}_{t+r}-\tilde{\mathcal{M}}_t)\Big\rangle\mid\mathcal{F}^{\tilde{W}}_T\vee\mathcal{F}^{\tilde{\mathcal{M}}}_t\Big]&=\exp\Big\{-\int_t^{t+r}\langle|\nabla\varphi|^2,\tilde{v}_s\rangle\mathrm{d}s\Big\}.\end{aligned}\tag{4.4}$$

Proof. Given a random variable Z which can be written as $g(\tilde{W})$, where g is bounded and continuous. We thus have

$$\begin{aligned}\tilde{\mathbb{E}}\Big[\exp\{i\langle\tilde{\mathcal{M}}_t,\varphi\rangle\}Z\Big]&=\lim_{N\to\infty}\tilde{\mathbb{E}}\Big[\exp\{i\langle\tilde{\mathcal{M}}^N_t,\varphi\rangle\}g(\tilde{W}^N)\Big]\\&=\lim_{N\to\infty}\mathbb{E}\Big[\exp\{i\langle\mathcal{M}^N_t,\varphi\rangle\}g(W)\Big]\\&=\lim_{N\to\infty}\mathbb{E}\Big[(\exp\{i\langle\mathcal{M}^N_t,\varphi\rangle\}-\exp\{i\langle\bar{\mathcal{M}}^N_t,\varphi\rangle\})g(W)\Big]\\&\quad+\lim_{N\to\infty}\mathbb{E}\Big[\exp\{i\langle\bar{\mathcal{M}}^N_t,\varphi\rangle\}g(W)\Big].\end{aligned}$$

where $\langle\bar{\mathcal{M}}^N_t,\varphi\rangle$ is defined as $\frac{\sqrt{2}}{\sqrt{N}}\sum_{i=1}^N\int_0^t\nabla\varphi(\bar{X}_i)\cdot\mathrm{d}B^i_s$, with X^N_i in $\mathcal{M}$ replaced by $\bar{X}_i$ from the conditional Mckean Vlasov equation (1.7).

Then,

$$\begin{aligned}&\mathbb{E}\Big|(\exp\{i\langle\mathcal{M}^N_t,\varphi\rangle\}-\exp\{i\langle\bar{\mathcal{M}}^N_t,\varphi\rangle\})g(W)\Big|\\&\lesssim\Big[\mathbb{E}\Big|\frac{\sqrt{2}}{\sqrt{N}}\sum_{i=1}^N\int_0^t\nabla\varphi(X^N_i)\cdot\mathrm{d}B^i_s-\frac{\sqrt{2}}{\sqrt{N}}\sum_{i=1}^N\int_0^t\nabla\varphi(\bar{X}_i)\cdot\mathrm{d}B^i_s\Big|^2\Big]^{\frac{1}{2}}\\&\lesssim\Big[\mathbb{E}\int_0^t|\nabla\varphi(X^N_1)-\nabla\varphi(\bar{X}_1)|^2\mathrm{d}s\Big]^{\frac{1}{2}}\end{aligned}$$

where the first inequality uses the fact that g is bounded and Hölder's inequality, while the second inequality follows from Burkholder-Davis-Gundy's inequality and the symmetry of the law of $(X^N,\bar{X}^N)$.

Using Proposition 4.1, we have

$$\lim_{N\to\infty}\mathbb{E}\Big|(\exp\{i\langle\mathcal{M}_t^N,\varphi\rangle\}-\exp\{i\langle\bar{\mathcal{M}}_t^N,\varphi\rangle\})g(W)\Big|=0.$$

Notice that $\{(\bar{X}_i,B_i)\}_{i\geqslant 1}$ is conditionally i.i.d. with respect to $\mathcal{F}_T^W$, we thus have

$$\begin{aligned}&\lim_{N\to\infty}\mathbb{E}\Big[\exp\{i\langle\bar{\mathcal{M}}_t^N,\varphi\rangle\}g(W)\Big]\\&=\lim_{N\to\infty}\mathbb{E}\Big[\mathbb{E}\Big[\exp\Big\{\frac{\sqrt{2}}{\sqrt{N}}\int_0^t\nabla\varphi(\bar{X}_1)\cdot\mathrm{d}B_s^1\Big\}\mid\mathcal{F}_T^W\Big]^N g(W)\Big]\\&=\mathbb{E}\Big[\exp\Big\{-\int_0^t\langle|\nabla\varphi|^2,v_s\rangle\mathrm{d}s\Big\}g(W)\Big]=\tilde{\mathbb{E}}\Big[\exp\Big\{-\int_0^t\langle|\nabla\varphi|^2,\tilde{v}_s\rangle\mathrm{d}s\Big\}Z\Big].\end{aligned}$$

where the second equality is a consequence of central limit theorem [DD19, Theorem 3.4.1]. By Lusin theorem, Z can be extended to be any bounded $\mathcal{F}_T^{\tilde{W}}$ measureable random variables. The proof of the second identity is similar. □

Before proceeding, let's recall the following abbreviation defined in Corollary 3.2, for $t\in[0,T]$ and $f\in L^2([0,T];H^4(\mathbb{T}^2))$,

$$\mathcal{R}_t(f)=\frac{1}{m}\exp\Big\{-\int_0^t m\|f_s\|_{H^4}^2\mathrm{d}s\Big\},\tag{4.5}$$

where the deterministic constant $m>1$ depends on $\|v_0\|_{L^2(\mathbb{T}^2)}$, $\inf_{x\in\mathbb{T}^2}v_0$ and T.

To prepare the proof for the convergence for the interacting term,

$$\int_0^t\tilde{\mathcal{K}}_s^N(\varphi)-\langle\tilde{v}_sK*\tilde{\eta}_s+\tilde{\eta}_sK*\tilde{v}_s,\nabla\varphi\rangle\mathrm{d}s\xrightarrow{N\to\infty}0,$$

we establish the following modified limit about fluctuation measures.

Lemma 4.3. *For every $\alpha>1$, it holds that*

$$\tilde{\mathbb{E}}\int_0^T\big\|\mathcal{R}_T(\tilde{v}^N)\tilde{\eta}_t^N-\mathcal{R}_T(\tilde{v})\tilde{\eta}_t\big\|_{H^{-\alpha}}\mathrm{d}t\xrightarrow{N\to\infty}0.\tag{4.6}$$

Proof. By Hölder's inequality, we have

$$\tilde{\mathbb{E}}\Big[\int_0^T\mathcal{R}_T(\tilde{v})\|\tilde{\eta}_t\|_{H^{-\alpha}}\mathrm{d}t\Big]^2\leqslant\sqrt{T}\Big[\tilde{\mathbb{E}}\int_0^T\mathcal{R}_T^2(\tilde{v})\|\tilde{\eta}_t\|_{H^{-\alpha}}^2\mathrm{d}t\Big]^{\frac{1}{2}}.$$

Recall the definitions about $\mathcal{R}_T(\tilde{v}^N),\mathcal{R}_T(\tilde{v})$ in (4.5) and notice that $\mathcal{R}_T^2(\tilde{v}^N)\leqslant\mathcal{R}_T(\tilde{v}^N)\leqslant\mathcal{R}_t(\tilde{v}^N)$, we then infer that $\forall\alpha>1$,

$$\begin{aligned}\tilde{\mathbb{E}}\int_0^T\mathcal{R}_T^2(\tilde{v})\|\tilde{\eta}_t\|_{H^{-\alpha}}^2\mathrm{d}t&\leqslant\sup_N\tilde{\mathbb{E}}\int_0^T\mathcal{R}_T(\tilde{v}^N)\|\tilde{\eta}_t^N\|_{H^{-\alpha}}^2\mathrm{d}t\\&\leqslant T\sup_{t\in[0,T]}\sup_N\tilde{\mathbb{E}}\Big[\mathcal{R}_T(\tilde{v}^N)\|\tilde{\eta}_t^N\|_{H^{-\alpha}}^2\Big]\\&\leqslant T\sup_{t\in[0,T]}\sup_N\mathbb{E}\Big[\mathcal{R}_t(v)\|\eta_t^N\|_{H^{-\alpha}}^2\Big]<\infty,\end{aligned}$$

where we used Corollary 3.2 to get the last inequality.

The above content provides the uniform integrability of

$$\int_0^T\big\|\mathcal{R}_T(\tilde{v}^N)\tilde{\eta}_t^N-\mathcal{R}_T(\tilde{v})\tilde{\eta}_t\big\|_{H^{-\alpha}}\mathrm{d}t.$$

Thus the convergence

$$\int_0^T \left\|\mathcal{R}_T(\tilde{v}^N)\tilde{\eta}_t^N - \mathcal{R}_T(\tilde{v})\tilde{\eta}_t\right\|_{H^{-\alpha}} \mathrm{d}t \xrightarrow{N\to\infty} 0, \quad \tilde{\mathbb{P}}-a.s.$$

leads to (4.6). □

We now deal with the interacting term.

Lemma 4.4. *For each $\varphi \in C^\infty(\mathbb{T}^2)$, it holds that*

$$\tilde{\mathbb{E}}\bigg(\sup_{t\in[0,T]}\bigg|\int_0^t \mathcal{R}_T(\tilde{v}^N)\tilde{\mathcal{K}}_s^N(\varphi) - \mathcal{R}_T(\tilde{v})\langle \tilde{v}_s K*\tilde{\eta}_s + \tilde{\eta}_s K*\tilde{v}_s, \nabla\varphi\rangle \mathrm{d}s\bigg|^{\frac{1}{2}}\bigg) \xrightarrow{N\to\infty} 0,$$

where

$$\tilde{\mathcal{K}}_t^N(\varphi) := \sqrt{N}\langle \nabla\varphi, K*\tilde{\mu}_N(t)\tilde{\mu}_N(t)\rangle - \sqrt{N}\langle\nabla\varphi, \tilde{v}_t^N K*\tilde{v}_t^N\rangle,$$

and

$$\tilde{\mu}_N := \frac{1}{\sqrt{N}}\tilde{\eta}^N + \tilde{v}^N.$$

Moreover, there exists a subsequence $\{N_k\}_{k\geqslant 1}$ (still denoted by $\{N\}_{N\in\mathbb{N}}$ for simplicity) such that

$$\int_0^t \tilde{\mathcal{K}}_s^N(\varphi) - \langle \tilde{v}_s K*\tilde{\eta}_s + \tilde{\eta}_s K*\tilde{v}_s, \nabla\varphi\rangle \mathrm{d}s \xrightarrow{N\to\infty} 0, \quad \forall t\in[0,T], \quad \tilde{\mathbb{P}}-a.s..$$

Proof. Notice that

$$\sqrt{N}\big(\tilde{\mu}_N K*\tilde{\mu}_N - \tilde{v}^N K*\tilde{v}^N\big) = \tilde{v}^N K*\tilde{\eta}^N + \tilde{\eta}^N K*\tilde{v}^N + \frac{1}{\sqrt{N}}\tilde{\eta}^N K*\tilde{\eta}^N.$$

Consequently, for each $\varphi\in C^\infty$, we have

$$\begin{aligned}
&\sup_{t\in[0,T]}\bigg|\int_0^t \mathcal{R}_T(\tilde{v}^N)\tilde{\mathcal{K}}_s^N(\varphi) - \mathcal{R}_T(\tilde{v})\langle \tilde{v}_s K*\tilde{\eta}_s + \tilde{\eta}_s K*\tilde{v}_s, \nabla\varphi\rangle\mathrm{d}s\bigg|^{\frac{1}{2}}\\
&\leqslant \sqrt{J_1^N(\varphi)} + \sqrt{J_2^N(\varphi)},
\end{aligned}\tag{4.7}$$

where

$$\begin{aligned}
J_1^N(\varphi) :=& \sqrt{N}\mathcal{R}_T(\tilde{v}^N)\int_0^T \big|\big\langle \nabla\varphi K*(\tilde{\mu}_N(t)-\tilde{v}_t^N), \tilde{\mu}_N(t)-\tilde{v}_t^N\big\rangle\big|\mathrm{d}t,\\
J_2^N(\varphi) :=& \int_0^T \big|\mathcal{R}_T(\tilde{v}^N)\big\langle \tilde{v}_t^N K*\tilde{\eta}_t^N + \tilde{\eta}_t^N K*\tilde{v}_t^N, \nabla\varphi\big\rangle\\
&- \mathcal{R}_T(\tilde{v})\big\langle \tilde{v}_t K*\tilde{\eta}_t + \tilde{\eta}_t K*\tilde{v}_t, \nabla\varphi\big\rangle\big|\mathrm{d}t.
\end{aligned}\tag{4.8}$$

On one hand, Lemma 3.2 yields the following estimate

$$\begin{aligned}
\tilde{\mathbb{E}}J_1^N(\varphi) &\leqslant T\sqrt{N}\sup_{t\in[0,T]}\tilde{\mathbb{E}}\big|\mathcal{R}_T(\tilde{v}^N)\langle\nabla\varphi K*(\tilde{\mu}_N(t)-\tilde{v}_t^N), \tilde{\mu}_N(t)-\tilde{v}_t^N\rangle\big|\\
&= T\sqrt{N}\sup_{t\in[0,T]}\mathbb{E}\big|\mathcal{R}_T(v)\langle\nabla\varphi K*(\mu_N(t)-v_t), \mu_N(t)-v_t\rangle\big|\\
&\leqslant T\sqrt{N}\sup_{t\in[0,T]}\mathbb{E}\big|\mathcal{R}_t(v)\langle\nabla\varphi K*(\mu_N(t)-v_t), \mu_N(t)-v_t\rangle\big| \lesssim N^{-\frac{1}{2}} \xrightarrow{N\to\infty} 0.
\end{aligned}$$

With Hölder's inequality, we infer $\tilde{\mathbb{E}}\sqrt{J_1^N(\varphi)} \leqslant \Big[\tilde{\mathbb{E}}(J_1^N(\varphi))\Big]^{\frac{1}{2}} \xrightarrow{N\to\infty} 0$. On the other hand, we have

$$\tilde{\mathbb{E}}\sqrt{J_2^N(\varphi)} \leqslant \tilde{\mathbb{E}}\Big[\sum_{i=1}^4\int_0^T H_i\mathrm{d}t\Big]^{\frac{1}{2}}.\tag{4.9}$$

where

$$\begin{aligned}H_1 :=&\big|\big\langle \tilde v_t K*\big(\mathcal{R}_T(\tilde v^N)\tilde\eta_t^N-\mathcal{R}_T(\tilde v)\tilde\eta_t\big),\nabla\varphi\big\rangle\big|,\\ H_2 :=&\big|\big\langle\big(\mathcal{R}_T(\tilde v^N)\tilde\eta_t^N-\mathcal{R}_T(\tilde v)\tilde\eta_t\big)K*\tilde v_t,\nabla\varphi\big\rangle\big|,\\ H_3 :=&\mathcal{R}_T(\tilde v^N)\big|\big\langle(\tilde v_t^N-\tilde v_t)K*\tilde\eta_t^N,\nabla\varphi\big\rangle\big|,\\ H_4 :=&\mathcal{R}_T(\tilde v^N)\big|\big\langle\tilde\eta_t^N K*(\tilde v_t^N-\tilde v_t),\nabla\varphi\big\rangle\big|.\end{aligned}$$

For each $t\in[0,T]$, we deduce that for every $\alpha>1$ that

$$\begin{aligned}H_1 =&\big|\big\langle K(-\cdot)*(\tilde v_t\nabla\varphi),\mathcal{R}_T(\tilde v^N)\tilde\eta_t^N-\mathcal{R}_T(\tilde v)\tilde\eta_t\big\rangle\big|\\ \leqslant&\big\|\mathcal{R}_T(\tilde v^N)\tilde\eta_t^N-\mathcal{R}_T(\tilde v)\tilde\eta_t\big\|_{H^{-\alpha}}\|K(-\cdot)*(\tilde v_t\nabla\varphi)\|_{H^\alpha},\end{aligned}$$

where

$$K(-\cdot)*g(x):=\int K(y-x)g(y)\mathrm{d}y. \tag{4.10}$$

Recall that $K\lesssim\frac{1}{|x|}\in L^1(\mathbb{T}^2)$, we apply Lemma 2.15 with $p=p_1=q=2$ and Lemma 2.14 to get

$$\begin{aligned}H_1 \lesssim&\ \big\|\mathcal{R}_T(\tilde v^N)\tilde\eta_t^N-\mathcal{R}_T(\tilde v)\tilde\eta_t\big\|_{H^{-\alpha}}\cdot\|K\|_{L^1}(\|\tilde v_t\|_{H^\alpha}\|\nabla\varphi\|_{L^\infty}+\|\tilde v_t\|_{L^\infty}\|\nabla\varphi\|_{H^\alpha}),\\ \lesssim&\ \big\|\mathcal{R}_T(\tilde v^N)\tilde\eta_t^N-\mathcal{R}_T(\tilde v)\tilde\eta_t\big\|_{H^{-\alpha}}\cdot\|K\|_{L^1}(\|\tilde v_t\|_{H^\alpha}\|\nabla\varphi\|_{L^\infty}+\|\tilde v_t\|_{H^2}\|\nabla\varphi\|_{H^\alpha}),\end{aligned}$$

where we use the Sobolev embedding theorem to get the last inequality.

Similarly, we have

$$\begin{aligned}H_2 \leqslant&\big\|\mathcal{R}_T(\tilde v^N)\tilde\eta_t^N-\mathcal{R}_T(\tilde v)\tilde\eta_t\big\|_{H^{-\alpha}}\|\nabla\varphi\cdot K*\tilde v_t\|_{H^\alpha}\\ \lesssim&\big\|\mathcal{R}_T(\tilde v^N)\tilde\eta_t^N-\mathcal{R}_T(\tilde v)\tilde\eta_t\big\|_{H^{-\alpha}}\cdot\|K\|_{L^1}(\|\tilde v_t\|_{H^\alpha}\|\nabla\varphi\|_{L^\infty}+\|\tilde v_t\|_{H^2}\|\nabla\varphi\|_{H^\alpha}),\end{aligned}$$

$$\begin{aligned}H_3 =&\ \mathcal{R}_T(\tilde v^N)\big|\big\langle K(-\cdot)*((\tilde v_t^N-\tilde v_t)\nabla\varphi),\tilde\eta_t^N\big\rangle\big|\\ \leqslant&\ \mathcal{R}_T(\tilde v^N)\|\tilde\eta_t^N\|_{H^{-\alpha}}\|K(-\cdot)*((\tilde v_t^N-\tilde v_t)\nabla\varphi)\|_{H^\alpha},\\ \lesssim&\ \mathcal{R}_T(\tilde v^N)\|\tilde\eta_t^N\|_{H^{-\alpha}}\cdot\|K\|_{L^1}(\|\tilde v_t^N-\tilde v_t\|_{H^\alpha}\|\nabla\varphi\|_{L^\infty}+\|\tilde v_t^N-\tilde v_t\|_{H^2}\|\nabla\varphi\|_{H^\alpha}),\end{aligned}$$

and

$$\begin{aligned}H_4 \leqslant&\ \mathcal{R}_T(\tilde v^N)\|\tilde\eta_t^N\|_{H^{-\alpha}}\|\nabla\varphi\cdot K*(\tilde v_t^N-\tilde v_t)\|_{H^\alpha}.\\ \lesssim&\ \mathcal{R}_T(\tilde v^N)\|\tilde\eta_t^N\|_{H^{-\alpha}}\cdot\|K\|_{L^1}(\|\tilde v_t^N-\tilde v_t\|_{H^\alpha}\|\nabla\varphi\|_{L^\infty}+\|\tilde v_t^N-\tilde v_t\|_{H^2}\|\nabla\varphi\|_{H^\alpha}).\end{aligned}$$

Next, we substitute these estimates into equation (4.9) with $\alpha=2$ and apply Hölder's inequality multiple times to obtain

$$\begin{aligned}\tilde{\mathbb{E}}\sqrt{J_2^N(\varphi)}\lesssim_\varphi&\ \|K\|_{L^1}^{\frac12}\tilde{\mathbb{E}}\Big[\int_0^T\mathcal{R}_T(\tilde v^N)\|\tilde\eta_t^N\|_{H^{-2}}\|\tilde v_t^N-\tilde v_t\|_{H^2}\mathrm{d}t\Big]^{\frac12}\\ &+\|K\|_{L^1}^{\frac12}\tilde{\mathbb{E}}\Big[\sup_{t\in[0,T]}\|\tilde v_t\|_{H^2}\int_0^T\big\|\mathcal{R}_T(\tilde v^N)\tilde\eta_t^N-\mathcal{R}_T(\tilde v)\tilde\eta_t\|_{H^{-2}}\mathrm{d}t\Big]^{\frac12}\\ \lesssim_\varphi&\ \|K\|_{L^1}^{\frac12}\tilde{\mathbb{E}}\Big(\Big[\int_0^T\mathcal{R}_T(\tilde v^N)\|\tilde\eta_t^N\|_{H^{-2}}^2\mathrm{d}t\Big]^{\frac14}\Big[\int_0^T\|\tilde v_t^N-\tilde v_t\|_{H^2}^2\mathrm{d}t\Big]^{\frac14}\Big)\\ &+\|K\|_{L^1}^{\frac12}\Big[\tilde{\mathbb{E}}\sup_{t\in[0,T]}\|\tilde v_t\|_{H^2}\Big]^{\frac12}\Big[\tilde{\mathbb{E}}\int_0^T\big\|\mathcal{R}_T(\tilde v^N)\tilde\eta_t^N-\mathcal{R}_T(\tilde v)\tilde\eta_t\big\|_{H^{-2}}\mathrm{d}t\Big]^{\frac12}\\ \lesssim_\varphi&\ \|K\|_{L^1}^{\frac12}\Big[T\sup_{t\in[0,T]}\tilde{\mathbb{E}}\mathcal{R}_T(\tilde v^N)\|\tilde\eta_t^N\|_{H^{-2}}^2\Big]^{\frac14}\Big[\tilde{\mathbb{E}}\Big[\int_0^T\|\tilde v_t^N-\tilde v_t\|_{H^2}^2\mathrm{d}t\Big]^{\frac12}\Big]^{\frac12}\\ &+\|K\|_{L^1}^{\frac12}\Big[\tilde{\mathbb{E}}\sup_{t\in[0,T]}\|\tilde v_t\|_{H^2}\Big]^{\frac12}\Big[\tilde{\mathbb{E}}\int_0^T\big\|\mathcal{R}_T(\tilde v^N)\tilde\eta_t^N-\mathcal{R}_T(\tilde v)\tilde\eta_t\big\|_{H^{-2}}\mathrm{d}t\Big]^{\frac12}.\end{aligned} \tag{4.11}$$

Observe that

$$\tilde{\mathbb{E}}\Big[\int_0^T \|\tilde{v}_t\|_{H^2}^2 \mathrm{d}t\Big] \lesssim \sup_N \tilde{\mathbb{E}}\Big[\int_0^T \|\tilde{v}_t^N\|_{H^2}^2 \mathrm{d}t\Big] = \mathbb{E}\Big[\int_0^T \|v_t\|_{H^2}^2 \mathrm{d}t\Big] < \infty,$$

which establishes the uniform integrability of

$$\Big[\int_0^T \|\tilde{v}_t^N - \tilde{v}_t\|_{H^2}^2 \mathrm{d}t\Big]^{\frac{1}{2}}.$$

Therefore, the convergence of

$$\int_0^T \|\tilde{v}_t^N - \tilde{v}_t\|_{H^2}^2 \mathrm{d}t, \quad \tilde{\mathbb{P}} - a.s.$$

together with Corollary 3.2, leads to the convergence of the first term in (4.11) to 0. Since $\sup_{t\in[0,T]} \|\cdot\|_{H^2}$ is lower semi-continuous on $\mathcal{V} = C([0,T]; L^2(\mathbb{T}^2)) \cap L^2([0,T]; H^4(\mathbb{T}^2))$, by Fatou's Lemma, we have

$$\tilde{\mathbb{E}} \sup_{t\in[0,T]} \|\tilde{v}_t\|_{H^2} \leqslant \liminf_{N\to\infty} \tilde{\mathbb{E}} \sup_{t\in[0,T]} \|\tilde{v}_t^N\|_{H^2} = \mathbb{E} \sup_{t\in[0,T]} \|v_t\|_{H^2} < \infty. \tag{4.12}$$

Thus, the second term in (4.11) converges to 0 by Lemma 4.3, completing the proof of the first claim.

Furthermore, the first claim implies that there exists a subsequence $\{N_k\}_{k\geqslant 1}$ (still denoted by $\{N\}_{N\in\mathbb{N}}$ for simplicity) such that

$$\int_0^t \mathcal{R}_T(\tilde{v}^N)\tilde{\mathcal{K}}_s^N(\varphi) - \mathcal{R}_T(\tilde{v})\langle \tilde{v}_s K * \tilde{\eta}_s + \tilde{\eta}_s K * \tilde{v}_s, \nabla\varphi\rangle \mathrm{d}s \xrightarrow{N\to\infty} 0, \quad \forall t \in [0,T], \quad \tilde{\mathbb{P}} - a.s..$$

Since $\mathcal{R}_T^{-1}(\tilde{v}^N) = m\exp\{\int_0^T m\|\tilde{v}_s^N\|_{H^4}^2 \mathrm{d}s\}$ converges to $\mathcal{R}_T^{-1}(\tilde{v}) = m\exp\{\int_0^T m\|\tilde{v}_s\|_{H^4}^2 \mathrm{d}s\}, \tilde{\mathbb{P}}$-a.s., we then conclude that

$$\int_0^t \tilde{\mathcal{K}}_s^N(\varphi) - \langle \tilde{v}_s K * \tilde{\eta}_s + \tilde{\eta}_s K * \tilde{v}_s, \nabla\varphi\rangle \mathrm{d}s \xrightarrow{N\to\infty} 0, \quad \forall t \in [0,T], \quad \tilde{\mathbb{P}} - a.s..$$

This completes the proof. □

Now, we are in the position to establish the following existence result, using the martingale approach as in [HRvR17, DHR21].

Theorem 4.5. $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\in[0,T]}, \tilde{\mathbb{P}}, \Big(\tilde{\eta}_t, \tilde{\mathcal{M}}_t, \tilde{W}_t\Big)_{t\in[0,T]})$ *is a probabilistically weak solution to the fluctuation SPDE* (1.11).

Firstly, we establish the convergence of the drift terms. As a consequence of Proposition 3.9, for every $\varphi \in C^\infty$, it holds $\tilde{\mathbb{P}}$-a.s. that for $\forall t \in [0,T]$,

$$\Big(\langle \tilde{\mathcal{M}}_t^N, \varphi\rangle), \langle \tilde{\eta}_t^N, \varphi\rangle, \langle \tilde{\eta}_0^N, \varphi\rangle, \int_0^t \langle \Delta\varphi, \tilde{\eta}_s^N\rangle \mathrm{d}s, \frac{1}{2}\int_0^t \Big\langle \sigma\cdot\nabla\Big(\sigma\cdot\nabla\varphi\Big), \tilde{\eta}_s^N\Big\rangle \mathrm{d}s\Big)$$

converges to

$$\Big(\langle \tilde{\mathcal{M}}_t, \varphi\rangle, \langle \tilde{\eta}_t, \varphi\rangle, \langle \tilde{\eta}_0, \varphi\rangle, \int_0^t \langle \Delta\varphi, \tilde{\eta}_s\rangle \mathrm{d}s, \frac{1}{2}\int_0^t \Big\langle \sigma\cdot\nabla\Big(\sigma\cdot\nabla\varphi\Big), \tilde{\eta}_s\Big\rangle \mathrm{d}s\Big).$$

Furthermore, recall Lemma 4.4, there exists a subsequence $\{N_k\}_{k\geqslant 1}$ (still denoted by $\{N\}_{N\in\mathbb{N}}$ for simplicity) such that

$$\int_0^t \tilde{\mathcal{K}}_s^N(\varphi) - \langle \tilde{v}_s K * \tilde{\eta}_s + \tilde{\eta}_s K * \tilde{v}_s, \nabla\varphi\rangle \mathrm{d}s \xrightarrow{N\to\infty} 0, \quad \forall t \in [0,T], \quad \tilde{\mathbb{P}} - a.s..$$

We then identify $\tilde{v}$ is the unique probabilistically strong solution to the mean field limit equation (1.5).

Proposition 4.6. *The process $\{\tilde W_t, t\in[0,T]\}$ is a 1-dimensional $(\tilde{\mathcal F}_t)_{t\in[0,T]}$ Brownian motion. Moreover, $\tilde v$ is the unique probabilistically strong solution to the mean field limit equation (1.5), in the sense defined by Definition 2.2 and 2.3.*

Proof. For $0\leqslant s<t\leqslant T$, let $\gamma:\mathcal H_s\to\mathbb R$ be a bounded and continuous function, where

$$\begin{aligned}\mathcal H_s :=&C([0,s];L^2)\times\cap_{k\in\mathbb N}C([0,s];H^{-3-\frac1k})\\ &\times\cap_{k\in\mathbb N}C([0,s];H^{-2-\frac1k})\times C([0,s];\mathbb R).\end{aligned}$$

We will use the abbreviations

$$\begin{aligned}\gamma^N :=&\gamma\left(v_{[0,s]},\eta^N_{[0,s]},\mathcal M^N_{[0,s]},W_{[0,s]}\right),\\ \tilde\gamma^N :=&\gamma\left(\tilde v^N_{[0,s]},\tilde\eta^N_{[0,s]},\tilde{\mathcal M}^N_{[0,s]},\tilde W^N_{[0,s]}\right),\\ \tilde\gamma :=&\gamma\left(\tilde v_{[0,s]},\tilde\eta_{[0,s]},\tilde{\mathcal M}_{[0,s]},\tilde W_{[0,s]}\right).\end{aligned}\tag{4.13}$$

Since the joint law of $(\tilde v^N,\tilde\eta^N,\tilde{\mathcal M}^N,\tilde W^N)$ coincides with the joint law of $(v,\eta^N,\mathcal M^N,W)$, we infer that

$$\begin{aligned}&\tilde{\mathbb E}\left(\tilde\gamma^N\left(\tilde W^N(t)-\tilde W^N(s)\right)\right)\\ =&\mathbb E\left(\gamma^N\left(W(t)-W(s)\right)\right)=0,\\ &\tilde{\mathbb E}\left(\tilde\gamma^N\tilde W^N(t)\tilde W^N(t)\right)-\tilde{\mathbb E}\left(\tilde\gamma^N\tilde W^N(s)\tilde W^N(s)\right)\\ =&\mathbb E\left(\gamma^N W(t)W(t)\right)-\mathbb E\left(\gamma^N W(s)W(s)\right)=t-s.\end{aligned}\tag{4.14}$$

Using Burkholder-Davis-Gundy inequality for W, we obtain the following uniform bound

$$\tilde{\mathbb E}|\tilde W^N(t)|^3=\mathbb E|W(t)|^3\leqslant Ct^{\frac32},\quad \forall N\in\mathbb N,$$

which provides the uniform integrability of the above terms. We thus can pass to the limit in the equations (4.14) and infer

$$\begin{aligned}&\tilde{\mathbb E}\left(\tilde\gamma\left(\tilde W(t)-\tilde W(s)\right)\right)=0,\\ &\tilde{\mathbb E}\left(\tilde\gamma\tilde W(t)\tilde W(t)\right)-\tilde{\mathbb E}\left(\tilde\gamma\tilde W(s)\tilde W(s)\right)=(t-s).\end{aligned}\tag{4.15}$$

This implies that $\{\tilde W_t,t\in[0,T]\}$ is a square-integrable $(\tilde{\mathcal F}_t)_{t\in[0,T]}$-martingale with $(\tilde{\mathcal F}_t)_{t\in[0,T]}$-quadratic variation $\Big[W,W\Big]_t=t$. By the Lévy martingale characterization theorem, we conclude that $\tilde W$ is a $(\tilde{\mathcal F}_t)_{t\in[0,T]}$ 1-dimensional Brownian motion.

Furthermore, as a consequence of Proposition 3.9, it holds $\tilde{\mathbb P}$-a.s. that $(\tilde v^N,\tilde W^N)$ with the same law $\mathcal L(v,W)$, where v is the unique probabilistically strong solution to the mean field limit equation (1.5) on the previous stochastic basis $(\Omega,\mathcal F,(\mathcal F^W_t)_{t\in[0,T]},\mathbb P,W)$, converges to $(\tilde v,\tilde W)$ in the space $C([0,T];L^2)\cap L^2([0,T];H^4),\tilde{\mathbb P}$-a.s.. This implies that $\mathcal L(\tilde v,\tilde W)=\mathcal L(v,W)$. Since we have established that $\tilde W$ is $(\tilde{\mathcal F}_t)_{t\in[0,T]}$-1-dimensional Brownian motion and given the well-posedness of the mean field limit equation (1.5) in Lemma 2.9, we conclude that $\tilde v$ is the unique probabilistically strong solution to (1.5). □

We now proceed to handle the transport noise term $\sigma\cdot\nabla\tilde\eta_t\mathrm d\tilde W_t$, using the approach in [HRvR17]. For $\varphi\in C^\infty$, let's define

$$\begin{aligned}\tilde Z_\cdot :=&\langle\tilde\eta_\cdot,\varphi\rangle-\langle\tilde\eta_0,\varphi\rangle-\int_0^\cdot\langle\Delta\varphi,\tilde\eta_s\rangle\mathrm ds-\int_0^\cdot\langle\nabla\varphi,v_sK*\tilde\eta_s\rangle\mathrm ds-\int_0^\cdot\langle\nabla\varphi,\tilde\eta_sK*v_s\rangle\mathrm ds\\ &-\mathcal M_\cdot(\varphi)-\frac12\int_0^\cdot\Big\langle\sigma\cdot\nabla\Big(\sigma\cdot\nabla\varphi\Big),\tilde\eta_s\Big\rangle\mathrm ds.\end{aligned}$$

(We do the same constructions to define $\tilde{Z}^N, Z^N$ for $\tilde{\eta}^N, \eta^N$).

Proposition 4.7. *The processes*

$$\tilde{Z}, \quad \tilde{Z}^2 - \int_0^t \langle \sigma \cdot \nabla\varphi, \tilde{\eta}_s \rangle^2 \, \mathrm{d}s, \quad \tilde{Z}\tilde{W} - \int_0^t \langle \sigma \cdot \nabla\varphi, \tilde{\eta}_s \rangle \, \mathrm{d}s,$$

indexed by $t \in [0, T]$, are $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}-$local-martingale.

Proof. We recall the following notations. For $0 \leqslant s < t \leqslant T$, $\gamma : \mathcal{H}_s \to \mathbb{R}$ denote a bounded and continuous function, where

$$\begin{aligned}\mathcal{H}_s :=& C([0, s]; L^2) \times \cap_{k\in\mathbb{N}} C([0, s]; H^{-3-\frac{1}{k}}) \\ & \times \cap_{k\in\mathbb{N}} C([0, s]; H^{-2-\frac{1}{k}}) \times C([0, s]; \mathbb{R}).\end{aligned}$$

We introduce the following abbreviations.

$$\begin{aligned}\gamma^N :=& \gamma\left(v_{[0,s]}, \eta^N_{[0,s]}, \mathcal{M}^N_{[0,s]}, W_{[0,s]}\right), \\ \tilde{\gamma}^N :=& \gamma\left(\tilde{v}^N_{[0,s]}, \tilde{\eta}^N_{[0,s]}, \tilde{\mathcal{M}}^N_{[0,s]}, \tilde{W}^N_{[0,s]}\right), \\ \tilde{\gamma} :=& \gamma\left(\tilde{v}_{[0,s]}, \tilde{\eta}_{[0,s]}, \tilde{\mathcal{M}}_{[0,s]}, \tilde{W}_{[0,s]}\right).\end{aligned}$$

Let $M > 0$ and define

$$\vartheta_M := C([0, T]; R) \to [0, T], \quad f \to \inf\{t > 0; |f(t)| \geqslant M\}$$

(with the convention $\inf \emptyset = T$). Choosing $\alpha_0 = \frac{7}{2}$ and noting that $\tilde{\eta}^N$ belongs to $C([0, T]; H^{-\alpha_0})$, we then deduce that for every $N \in \mathbb{N}$, $\vartheta_M(\tilde{I}^N)$ defines an $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$-stopping time and the blow up does not occur in a finite time, i.e.

$$\sup_{M>0} \vartheta_M(\tilde{I}^N) = T \quad \tilde{\mathbb{P}} - a.s., \tag{4.16}$$

where $\tilde{I}^N(t) := \sup_{s\in[0,t]} \|\tilde{\eta}^N\|_{H^{-\alpha_0}}$. The same is valid for the case $\tilde{I}(t) := \sup_{s\in[0,t]} \|\tilde{\eta}\|_{H^{-\alpha_0}}$. The stopping times $\vartheta_M(\tilde{I})$ will play the role of a localizing sequence for the processes

$$\tilde{Z}, \quad \tilde{Z}^2 - \int_0^t \langle \sigma \cdot \nabla\varphi, \tilde{\eta}_s \rangle^2 \, \mathrm{d}s, \quad \tilde{Z}\tilde{W} - \int_0^t \langle \sigma \cdot \nabla\varphi, \tilde{\eta}_s \rangle \, \mathrm{d}s.$$

Due to the observation made in [HS13, Lemma 3.5, Lemma 3.6], there exists a sequence $\{M_n\} \to \infty$ such that

$$\tilde{\mathbb{P}}\Big(\vartheta_{M_n}(\cdot) \text{ is continuous at } \tilde{I}\Big) = 1.$$

Consequencely, we establish the convergence of stopping times, that is, for fixed $n \in \mathbb{N}$,

$$\vartheta_{M_n}(\tilde{I}^N) \xrightarrow{N\to\infty} \vartheta_{M_n}(\tilde{I}), \quad \tilde{\mathbb{P}} - a.s..$$

Since the joint law of $(\tilde{v}^N, \tilde{\eta}^N, \tilde{\mathcal{M}}^N, \tilde{W}^N)$ coincides with the joint law of $(v, \eta^N, \mathcal{M}^N, W)$, we then have for every $n \in \mathbb{N}$ and $0 \leqslant s < t \leqslant T$,

$$\begin{aligned}&\tilde{\mathbb{E}}\Big[\tilde{\gamma}^N \tilde{Z}^N(t \wedge \vartheta_{M_n}(\tilde{I}^N))\Big] = \tilde{\mathbb{E}}\Big[\tilde{\gamma}^N \tilde{Z}^N(s \wedge \vartheta_{M_n}(\tilde{I}^N))\Big], \\ &\tilde{\mathbb{E}}\Big[\tilde{\gamma}^N\Big((\tilde{Z}^N(t \wedge \vartheta_{M_n}(\tilde{I}^N))^2 - \int_0^{t\wedge\vartheta_{M_n}(\tilde{I}^N)} \left\langle \sigma \cdot \nabla\varphi, \tilde{\eta}^N_r \right\rangle^2 \mathrm{d}r\Big)\Big] \\ =&\tilde{\mathbb{E}}\Big[\tilde{\gamma}^N\Big((\tilde{Z}^N(s \wedge \tau_{M_n}(\tilde{I}^N)))^2 - \int_0^{s\wedge\tau_{M_n}(\tilde{I}^N)} \left\langle \sigma \cdot \nabla\varphi, \tilde{\eta}^N_r \right\rangle^2 \mathrm{d}r\Big)\Big],\end{aligned}$$

and

$$\tilde{\mathbb{E}}\Big[\tilde{\gamma}^N\Big(\tilde{W}^N(t\wedge\vartheta_{M_n}(\tilde{I}^N))\tilde{Z}^N(t\wedge\vartheta_{M_n}(\tilde{I}^N))-\int_0^{t\wedge\vartheta_{M_n}(\tilde{I}^N)}\big\langle\sigma\cdot\nabla\varphi,\tilde{\eta}^N_r\big\rangle\,\mathrm{d}r\Big)\Big]$$
$$=\tilde{\mathbb{E}}\Big[\tilde{\gamma}^N\Big(\tilde{W}^N(s\wedge\vartheta_{M_n}(\tilde{I}^N))\tilde{Z}^N(s\wedge\vartheta_{M_n}(\tilde{I}^N))-\int_0^{s\wedge\vartheta_{M_n}(\tilde{I}^N)}\big\langle\sigma\cdot\nabla\varphi,\tilde{\eta}^N_r\big\rangle\,\mathrm{d}r\Big)\Big].$$

The Burkholder-Davis-Gundy's inequality for $\tilde{Z}^N(t\wedge\vartheta_{M_n}(\tilde{I}^N))$ yields the uniform bound

$$\tilde{\mathbb{E}}\big|\tilde{Z}^N(t\wedge\vartheta_{M_n}(\tilde{I}^N))\big|^4=\mathbb{E}\Big|\int_0^{t\wedge\vartheta_{M_n}(I^N)}\big\langle\sigma\cdot\nabla\varphi,\tilde{\eta}^N_r\big\rangle\,\mathrm{d}W_r\Big|^4$$
$$\leqslant\mathbb{E}\Big|\int_0^{t\wedge\vartheta_{M_n}(I^N)}\big\langle\sigma\cdot\nabla\varphi,\tilde{\eta}^N_r\big\rangle^2\,\mathrm{d}r\Big|^2\leqslant C_{M_n},\quad\forall N\in\mathbb{N},$$

which provide the necessary uniform integrability. We thus can pass the limit in equations and infer

$$\tilde{\mathbb{E}}\Big[\tilde{\gamma}\tilde{Z}(t\wedge\vartheta_{M_n}(\tilde{I}))\Big]=\tilde{\mathbb{E}}\Big[\tilde{\gamma}\tilde{Z}(s\wedge\vartheta_{M_n}(\tilde{I}))\Big],$$
$$\tilde{\mathbb{E}}\Big[\tilde{\gamma}\Big(\tilde{Z}^2(t\wedge\vartheta_{M_n}(\tilde{I}))-\int_0^{t\wedge\vartheta_{M_n}(\tilde{I})}\langle\sigma\cdot\nabla\varphi,\tilde{\eta}_r\rangle^2\,\mathrm{d}r\Big)\Big]$$
$$=\tilde{\mathbb{E}}\Big[\tilde{\gamma}\Big(\tilde{Z}^2(s\wedge\vartheta_{M_n}(\tilde{I}))-\int_0^{s\wedge\vartheta_{M_n}(\tilde{I})}\langle\sigma\cdot\nabla\varphi,\tilde{\eta}_r\rangle^2\,\mathrm{d}r\Big)\Big],$$

and

$$\tilde{\mathbb{E}}\Big[\tilde{\gamma}\Big(\tilde{W}(t\wedge\vartheta_{M_n}(\tilde{I}))\tilde{Z}(t\wedge\vartheta_{M_n}(\tilde{I}))-\int_0^{t\wedge\vartheta_{M_n}(\tilde{I})}\langle\sigma\cdot\nabla\varphi,\tilde{\eta}_r\rangle\,\mathrm{d}r\Big)\Big]$$
$$=\tilde{\mathbb{E}}\Big[\tilde{\gamma}\Big(\tilde{W}(s\wedge\vartheta_{M_n}(\tilde{I}))\tilde{Z}(s\wedge\vartheta_{M_n}(\tilde{I}))-\int_0^{s\wedge\vartheta_{M_n}(\tilde{I})}\langle\sigma\cdot\nabla\varphi,\tilde{\eta}_r\rangle\,\mathrm{d}r\Big)\Big].$$

Therefore, for every $n\in\mathbb{N}$,

$$\tilde{Z}(\cdot\wedge\vartheta_{M_n}(\tilde{I})),$$
$$\tilde{Z}^2(\cdot\wedge\vartheta_{M_n}(\tilde{I}))-\int_0^{\cdot\wedge\vartheta_{M_n}(\tilde{I})}\langle\sigma\cdot\nabla\varphi,\tilde{\eta}_r\rangle^2\,\mathrm{d}r,$$
$$\tilde{W}(\cdot\wedge\vartheta_{M_n}(\tilde{I}))\tilde{Z}(\cdot\wedge\vartheta_{M_n}(\tilde{I}))-\int_0^{\cdot\wedge\vartheta_{M_n}(\tilde{I})}\langle\sigma\cdot\nabla\varphi,\tilde{\eta}_r\rangle\,\mathrm{d}r,$$

are $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}-$martingales, which completes the proof. □

Proof of Theorem 4.5. Having Proposition 4.6 and Proposition 4.7 in hand, we can directly calculate

$$\Big[\tilde{Z}(\cdot\wedge\vartheta_{M_n}(\tilde{I}))-\int_0^{\cdot\wedge\vartheta_{M_n}(\tilde{I})}\langle\sigma\cdot\nabla\varphi,\tilde{\eta}_r\rangle\,\mathrm{d}W_r,\tilde{Z}(\cdot\wedge\vartheta_{M_n}(\tilde{I}))-\int_0^{\cdot\wedge\vartheta_{M_n}(\tilde{I})}\langle\sigma\cdot\nabla\varphi,\tilde{\eta}_r\rangle\,\mathrm{d}W_r\Big]_t=0$$

for every $n\in\mathbb{N}$. Let's $n\to\infty$, the fluctuation SPDE (2.5) holds. Combining the results of Lemma 4.2 and Proposition 4.6, we complete the proof. □

4.2. **Uniqueness.** In this subsection, we demonstrate pathwise uniqueness for the fluctuation SPDE (1.11) and complete the proof of our main results, Theorem 1.2 and Theorem 1.1.

Theorem 4.8. *Pathwise uniqueness holds true for the fluctuation SPDE* (1.11) *in the sense of Deifinition* (2.7).

Proof. Let $W_t, t \in [0,T]$ be a 1-dimensional Brownian motion and stochastic process $(\mathcal{M}_t)_{t\in[0,T]}$ takes values in $\bigcap_{k\in\mathbb{N}} C([0,T]; H^{-2-\frac{1}{k}}(\mathbb{T}^d))$, both defined on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P})$. Suppose further that

$$\left(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{P}, \left(\eta^i_t, \mathcal{M}_t, W_t\right)_{t\in[0,T]}\right)_{i=1,2}$$

are probabilistically weak solution to (1.11) with the same initial data η_0. We define $\bar{\eta} := \eta^1 - \eta^2$ and obtain that $\forall \varphi \in C^\infty(\mathbb{T}^2)$,

$$\begin{aligned}\langle \bar{\eta}_t, \varphi\rangle &= \int_0^t \langle \Delta\varphi, \bar{\eta}_s\rangle \mathrm{d}s + \int_0^t \langle \nabla\varphi, v_s K * \bar{\eta}_s\rangle \mathrm{d}s + \int_0^t \langle \nabla\varphi, \bar{\eta}_s K * v_s\rangle \mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \left\langle \sigma\cdot\nabla\Big(\sigma\cdot\nabla\varphi\Big), \bar{\eta}_s\right\rangle \mathrm{d}s + \int_0^t \langle \sigma\cdot\nabla\varphi, \bar{\eta}_s\rangle \,\mathrm{d}W_s, \quad \forall t\in[0,T], \mathbb{P}-a.s..\end{aligned}$$

Then, we evolve $|\langle \bar{\eta}_\cdot, \varphi\rangle|^2$ for each e_k of the Fourier basis $\{e_k := e^{\sqrt{-1}k\cdot x}, k \in \mathbb{Z}^2\}$, using Itô's formula.

$$\begin{aligned}\mathrm{d}\mid \langle \bar{\eta}_t, e_k\rangle \mid^2 &= \langle \bar{\eta}_t, e_k\rangle \Big[\langle \Delta e_{-k}, \bar{\eta}_t\rangle + \langle \nabla e_{-k}, v_t K * \bar{\eta}_t\rangle \\ &\quad + \langle \nabla e_{-k}, \bar{\eta}_t K * v_t\rangle + \frac{1}{2}\left\langle \sigma\cdot\nabla\Big(\sigma\cdot\nabla e_{-k}\Big), \bar{\eta}_t\right\rangle\Big]\mathrm{d}t \\ &\quad + \langle \bar{\eta}_t, e_{-k}\rangle \Big[\langle \Delta e_k, \bar{\eta}_t\rangle + \langle \nabla e_k, v_t K * \bar{\eta}_t\rangle \\ &\quad + \langle \nabla e_k, \bar{\eta}_t K * v_t\rangle + \frac{1}{2}\left\langle \sigma\cdot\nabla\Big(\sigma\cdot\nabla e_k\Big), \bar{\eta}_t\right\rangle\Big]\mathrm{d}t \\ &\quad + \langle \bar{\eta}_t, e_{-k}\rangle \langle \sigma\cdot\nabla e_k, \bar{\eta}_t\rangle \,\mathrm{d}W_t + \langle \bar{\eta}_t, e_k\rangle \langle \sigma\cdot\nabla e_{-k}, \bar{\eta}_t\rangle \,\mathrm{d}W_t \\ &\quad + \langle \sigma\cdot\nabla e_{-k}, \bar{\eta}_t\rangle \langle \sigma\cdot\nabla e_k, \bar{\eta}_t\rangle \,\mathrm{d}t.\end{aligned}$$

For fixed $3 > \alpha > 2$, we now sum $(1+|k|^2)^{-\alpha-1} \mid \langle \bar{\eta}_t, e_k\rangle \mid^2$ over $k \in \mathbb{Z}^2$, and obtain

$$\|\bar{\eta}_t\|^2_{H^{-\alpha-1}} = \sum_{i=1}^3 J_i(t) + L_t,$$

where

$$\begin{aligned}J_1(t) &:= -2\int_0^t \sum_{k\in\mathbb{Z}^2} |k|^2(1+|k|^2)^{-\alpha-1} \mid \langle \bar{\eta}_s, e_k\rangle \mid^2 \,\mathrm{d}s, \\ J_2(t) &:= \sum_{k\in\mathbb{Z}^2} (1+|k|^2)^{-\alpha-1} \int_0^t \langle \bar{\eta}_s, e_{-k}\rangle \Big[\sqrt{-1}k\cdot\langle K * \bar{\eta}_s v_s, e_k\rangle \\ &\quad + \sqrt{-1}k\cdot\langle K * v_s \bar{\eta}_s, e_k\rangle\Big] + \langle \bar{\eta}_s, e_k\rangle \Big[-\sqrt{-1}k\cdot\langle K * \bar{\eta}_s v_s, e_{-k}\rangle \\ &\quad - \sqrt{-1}k\cdot\langle K * v_s \bar{\eta}_s, e_{-k}\rangle\Big]\mathrm{d}s, \\ J_3(t) &:= \int_0^t \|\sigma\cdot\nabla\bar{\eta}_s\|^2_{H^{-\alpha-1}} + \langle \sigma\cdot\nabla(\sigma\cdot\nabla\bar{\eta}_s), \bar{\eta}_s\rangle_{H^{-\alpha-1}} \,\mathrm{d}s,\end{aligned}$$

and $L_t := \int_0^t \langle \sigma\cdot\nabla\bar{\eta}_s, \bar{\eta}_s\rangle_{H^{-\alpha-1}} \,\mathrm{d}W_s$ is a continuous local martingale.

Applying Young's inequality, we find for every $\varepsilon > 0$, there exists a constant C_ε such that

$$\int_0^t \sqrt{-1}k\cdot\langle \bar{\eta}_s, e_{-k}\rangle \langle K * \bar{\eta}_s v_s, e_k\rangle \mathrm{d}s \leqslant \varepsilon \int_0^t |k|^2 |\langle \bar{\eta}_s, e_{-k}\rangle|^2 \mathrm{d}s + C_\varepsilon \int_0^t |\langle K * \bar{\eta}_s v_s, e_k\rangle|^2 \mathrm{d}s.$$

Consequently, we have

$$\int_0^t \sum_{k\in\mathbb{Z}^2} (1+|k|^2)^{-\alpha-1} \sqrt{-1}k\cdot\langle \bar{\eta}_s, e_{-k}\rangle \langle K * \bar{\eta}_s v_s, e_k\rangle \mathrm{d}s$$

$$\leqslant \varepsilon \int_0^t \|\eta_s\|_{H^{-\alpha}}^2 \mathrm{d}s + C_\varepsilon \int_0^t \|K * \bar{\eta}_s v_s\|_{H^{-\alpha-1}}^2 \mathrm{d}s.$$

By Lemma 2.12, Lemma 2.13 and [BFH18, Theorem A.1.3], for $2 < \alpha < 3$, we know

$$\begin{aligned}\|K * \bar{\eta}_s v_s\|_{H^{-\alpha-1}} &\leqslant \|K * \bar{\eta}_s v_s\|_{H^{-\alpha}} \\ &\lesssim \|K * \bar{\eta}_s\|_{H^{-\alpha}} \|v_s\|_{H^4} \\ &= \|\nabla^\perp (-\Delta)^{-1} \bar{\eta}_s\|_{H^{-\alpha}} \|v_s\|_{H^4} \\ &\lesssim \|\bar{\eta}_s\|_{H^{-\alpha-1}} \|v_s\|_{H^4}.\end{aligned}$$

Here we used the fact Biot-Savart kernel $K = \nabla^\perp G$, where G is the Green function of $-\Delta$ on torus, as given in [FGP11]. We thus conclude that

$$\int_0^t \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-\alpha-1} \sqrt{-1} k \cdot \langle \bar{\eta}_s, e_{-k}\rangle \langle K * \bar{\eta}_s v_s, e_k\rangle \mathrm{d}s \leqslant \varepsilon \int_0^t \|\eta_s\|_{H^{-\alpha}}^2 \mathrm{d}s + C_\varepsilon \int_0^t \|v_s\|_{H^4}^2 \|\bar{\eta}\|_{H^{-\alpha-1}}^2 \mathrm{d}s.$$

The other three terms in $J_2(t)$ can be controlled in a similar way. We then obtain that for every $\varepsilon > 0$, there exists a constant C_ε such that for $2 < \alpha < 3$,

$$J_2(t) \leqslant \varepsilon \int_0^t \|\eta_s\|_{H^{-\alpha}}^2 \mathrm{d}s + C_\varepsilon \int_0^t \|v_s\|_{H^4}^2 \|\bar{\eta}\|_{H^{-\alpha-1}}^2 \mathrm{d}s.$$

Choosing $\varepsilon < 2$, we conclude that for $2 < \alpha < 3$,

$$\begin{aligned}J_1(t) + J_2(t) &= -2 \int_0^t \sum_{k\in\mathbb{Z}^2} (1+|k|^2)^{-\alpha} \mid \langle \bar{\eta}_s, e_k\rangle \mid^2 \mathrm{d}s + 2\int_0^t \sum_{k\in\mathbb{Z}^2} (1+|k|^2)^{-\alpha-1} \mid \langle \bar{\eta}_s, e_k\rangle \mid^2 \mathrm{d}s + J_2(t) \\ &\leqslant (\varepsilon - 2) \int_0^t \|\eta_s\|_{H^{-\alpha}}^2 \mathrm{d}s + C_\varepsilon \int_0^t \|v_s\|_{H^4}^2 \|\bar{\eta}\|_{H^{-\alpha-1}}^2 \mathrm{d}s + 2\int_0^t \|\bar{\eta}\|_{H^{-\alpha-1}}^2 \mathrm{d}s \\ &\lesssim \int_0^t (\|v_s\|_{H^4}^2 + 1) \|\bar{\eta}\|_{H^{-\alpha-1}}^2 \mathrm{d}s.\end{aligned}$$

Furthermore, applying [Kry15, Lemma 2.3] about commutator estimate yields

$$J_3(t) \leqslant C_\sigma \int_0^t \|\bar{\eta}\|_{H^{-\alpha-1}}^2 \mathrm{d}s.$$

Consequently, we finally have

$$\|\bar{\eta}_t\|_{H^{-\alpha-1}}^2 \lesssim \int_0^t (\|v_s\|_{H^4}^2 + 1) \|\bar{\eta}\|_{H^{-\alpha-1}}^2 \mathrm{d}s + L_t.$$

Using stochastic Gronwall's inequality in [Gei21, Corollary 5.4], we have $\sup_{t\in[0,T]} \|\bar{\eta}_t\|_{H^{-\alpha-1}}^2 = 0$ $\mathbb{P}$-a.s.. The proof is then completed. □

Proof of Theorem 1.1. Having established the existence of probabilistically weak solution to the fluctuation SPDE (1.11) and the pathwise uniqueness for the fluctuation SPDE (1.11) given in Theorem 4.5 and Theorem 4.8 in hand, we apply the general Yamada-Watanabe theorem [Kur14, Theorem 1.5] to conclude the well-posedness of (1.11). We have shown in Lemma 3.8 that the sequence of laws of $\{\eta^N\}_{N\in\mathbb{N}}$ is tight, and in Theorem 4.5 that every limiting point is a probabilistically weak solution to (1.11). Through the well-posedness of (1.11) and general Yamada-Watanabe theorem [Kur14, Theorem 1.5], we then conclude that every limiting point is the unique probabilistically strong solution to (1.11) and the law of every limiting point is unique. This establishes the convergence of the fluctuation measures $(\eta^N)_{N\geqslant 1}$ to the fluctuation SPDE (1.11), completing the proof.

□

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