STATIONARY MEAN-FIELD GAMES OF SINGULAR CONTROL UNDER KNIGHTIAN UNCERTAINTY

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ABSTRACT. In this work, we study a class of stationary mean-field games of singular stochastic control under model uncertainty. The representative agent adjusts the dynamics of an Itô-diffusion via one-sided singular stochastic control, aiming to maximize a long-term average expected profit criterion. The mean-field interaction is of scalar type through the stationary distribution of the population. Due to the presence of uncertainty, the problem involves the study of a stochastic (zero-sum) game, where the decision maker chooses the 'best' singular control policy, while the adversarial player selects the 'worst' probability measure. Using a constructive approach, we prove existence and uniqueness of a stationary mean-field equilibrium. Finally, we present an example of mean-field optimal extraction of natural resources under uncertainty and we analyze the impact of uncertainty on the mean-field equilibrium.

Keywords: stationary mean-field games; singular control; model uncertainty; ergodic criterion; free-boundary problem; shooting method.

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1. INTRODUCTION

Mean-field games (MFGs in short) were independently introduced in 2006 by Lasry and Lions [52], and by Caines et al. [12], as asymptotic models for symmetric N-player differential games. In these settings, each player's dynamics and decisions are influenced by the collective behavior of the population, typically represented by the empirical distribution of the states (and potentially the actions) of all players (extended MFGs). The central idea of MFGs is to replace the complex many-player interaction with the problem of a single representative agent who optimizes her strategy in response to a given flow of probability measures, reflecting the statistical distribution of the other, indistinguishable agents. The equilibrium concept in MFGs emerges as a consistency condition: The law of the optimally controlled state process of the representative agent must coincide with the prescribed flow of distributions. In this way, the classical Nash equilibrium from the N-player game is replaced by a fixed-point requirement on the evolution of distributions. Since their introduction, MFGs have garnered substantial attention in both the mathematical and applied communities. This is due to their analytical tractability, their deep connections with the theory of propagation of chaos and forward-backward stochastic systems, and their capacity to approximate ε_N-Nash equilibria in large, symmetric N-player games. For a comprehensive treatment of the theory, methodologies, and main results, we refer the reader to the two-volume monograph by Carmona and Delarue [21]. A detailed overview of applications of MFGs in fields such as Economics, Finance, and Engineering can be found in the survey by Carmona [20].

In stationary MFGs, the representative player interacts with the long-run distribution of the population. Such a concept has a long tradition in economic theory: Stationary equilibria appeared already in the 1980s in the context of games with a continuum of players (see [44] and [47]), and also play an important role in the analysis of competitive market models with heterogeneous agents (see, e.g., [1]

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and [54], amongst many others). Closely connected is also the concept of stationary oblivious equilibria, introduced by Adlakha et al. in [2]. Within the mathematical literature, stationary MFGs have been approached both via analytic and probabilistic methods. Among those papers adopting a partial differential equations (PDE) approach, we refer to the works of Bardi and Feleqi [6] for the study of the forward-backward system arising in stationary MFGs with regular controls, Gomes et al. [41] for extended stationary MFGs, Cardaliaguet and Porretta [19] for the study of the long-term behavior of the master equation arising in MFG theory, and to Bertucci [9] for the study of stationary mean-field optimal stopping games. On the other hand, a probabilistic approach is followed in a series of recent contributions dealing with stationary MFGs with singular and impulsive controls, see Aïd et al. [3], Cannerozzi and Ferrari [16], Cao and Guo [18], Cao et al. [17] and Dianetti et. al. [33].

The aforementioned problems are based on the assumption that agents possess complete certainty regarding the occurrence of system events-that is, the real-world probability measure is perfectly known to them. However, this assumption is unrealistic, as economic and financial models often involve complex mechanisms and multiple sources of uncertainty. A well-known concept that addresses this issue is Knightian uncertainty [49] (also referred to as model uncertainty), which describes situations in which the decision maker has incomplete knowledge about the probabilities of various outcomes. To account for this, the concept of ambiguity has been introduced, wherein the decision maker evaluates her objective function by minimizing it over a set of plausible probability measures, commonly referred to as the set of priors. In this context, Gilboa and Schmeidler [40] proposed the max-min expected utility framework, in which the agent maximizes expected utility with respect to the worst-case prior within a suitable set. This approach was further extended by Hansen and Sargent [43], who developed a continuous-time version of the max-min expected utility framework and explored its connection to robust control theory. Over the past two decades, optimization problems under model uncertainty have played a significant role in economics and finance. A detailed literature review falls outside the scope of this paper. We only would like to mention papers that deal with optimal timing and singular control problems under uncertainty. Among these, optimal stopping problems under ambiguity have been studied by Nutz and Zhang [56], Riedel [61], and Riedel and Cheng [23], among others. For singular control problems, we refer to the works of Chakraborty et al. [22], Cohen [26], Cohen et al. [27] Ferrari et. al. [37] and Ferrari et al. [38], while Perninge [57] examines impulse control problems under model uncertainty.

1.1. **Our Results.** In this paper, we study a class of stationary MFGs under model uncertainty, where the underlying state process is a general singularly controlled one-dimensional diffusion. More precisely, the representative agent optimally controls a real-valued Itô-diffusion through a one-sided singular control in order to maximize an ergodic reward functional while is uncertain about the *real-world model*. To take into account *model uncertainty*, agent maximizes a long-time-average of the time-integral of a running profit function, net of the proportional costs of actions under the worst-case scenario probability measure. The latter can be addressed as a zero-sum game between the agent and an *adverse player* (see, for instance, Cohen et. al. [27]). The mean-field interaction is of scalar type and comes through a real-valued parameter denoted by θ, which, at equilibrium, has to identify with a suitable generalized moment of the stationary distribution of the optimally controlled state process. From the economic point of view, θ can be thought of as a stationary price index arising from the aggregate productivity through an isoelastic demand function à la Spence-Dixit-Stiglitz (see pp. 7-8 in [1]). We refer to Remark 2.2 below for details.

Our first contribution is the solution of the representative player's optimal control problem. In this context, we extend the result of Cohen et al. [27] to a setting that includes running profit and state-dependent proportional cost of control (see Theorem 3.2 below). This is achieved by applying the *shooting method*, following an approach similar to that used in [27]. Using a verification argument (see Proposition 3.1 and Theorem 3.2), we demonstrate that, for a fixed mean-field parameter, θ, the optimal control is of barrier-type. That is, the optimal control uniquely solves a Skorokhod reflection problem (see e.g., Tanaka [63]) at endogenously determined barrier, which depend on the level of ambiguity and on the given and fixed mean-field parameter θ.

The next step deals with the construction of the MFG equilibrium and with the proof of its uniqueness. To that end, we first show that the process constituted by the optimally controlled diffusion process under the worst-case scenario admits a stationary distribution (cf. Proposition 4.2 below) and the stationary density function has an explicit form (cf. (4.32)). Clearly, the stationary distribution and its density function depend on the level of ambiguity and fixed mean-field parameter θ, since the optimally controlled state does. In order to proceed with the equilibrium analysis, we thus study the stability of the stationary distribution with respect to θ and actually prove its continuity with respect to such a parameter (cf. Proposition 4.2). Further exploiting the connection to the auxiliary boundary value problem, we are then able to show the monotonicity and the boundedness from below of the free-boundary with respect to mean-field parameter (cf. Lemmata 4.1 and 4.2 below) and to determine an invariant compact set where any equilibrium value of θ (if one exists) should lie. Combining those continuity and compactness results, an application of the Schauder-Tychonof fixed point theorem allows us to prove that there exists a stationary equilibrium (cf. Theorem 4.2), which is then also proved to be unique.

Finally, we complement our theoretical analysis by a case study arising in the context of optimal extraction model. Here, we assume that the representative firm extracts from a natural resource which evolves as an affine diffusion process with mean-reverting drift and the profit function is of power type. In this setting, we study the sensitivity of equilibria on level of ambiguity and the level of volatility. Due to the presence of a quadratic term in the variational inequality (cf. (3.5)), the problem cannot be solved explicitly. Therefore, we develop a policy iteration algorithm (PIA) to approximate the equilibria.

1.2. **Related Literature.** Ergodic singular stochastic control problems for one-dimensional diffusions have been treated in general settings, including state-dependent costs of actions, and with different applications; see [5], [53], [55] and [46], [50], among others. However, in all those papers, model uncertainty is not considered.

Our paper is placed within the recent bunch of literature dealing with MFGs with singular controls by following a probabilistic approach; see Aïd et al. [3], Cao and Guo [18], Cao et al. [17], Campi et al. [15], Cohen and Sun [28], Dianetti et al. [32], Dianetti et. al. [33], Denkert and Horst [31], Fu and Horst [39], and Guo and Xu [42]. Amongst those, the work that most relates to ours is by Cao et al. [17]. Cao et al. consider in [17] ergodic MFGs involving a one-dimensional singularly controlled Itô-diffusion that can be increased via a monotone control process. In contrast to our work, [17] does not consider model uncertainty. Our paper is also closely related to the works of Chakraborty et al. [22], Cohen [26], and Cohen et al. [27]. In particular, like these studies, we consider worst-case scenarios modeled via Kullback-Leibler divergence and employ the *shooting method* to solve the stochastic singular control problem, following the approach in [27]. However, these papers do not incorporate a mean-field game framework.

We also clearly relate to those works dealing with MFGs involving model uncertainty. Huang and Huang [45] consider mean-field linear-quadratic-Gaussian control under model uncertainty. Bauso et. al. in [7] focus on robust MFGs and risk-sensitive type MFGs. Furthermore, Langner et. al. [51] study Markov-Nash equilibrium (discrete time and state space) under model uncertainty.

1.3. **Organization of the Paper.** The rest of the paper is organized as follows. In Section 2, we introduce the probabilistic setting and the MFG under study. Next, in Section 3, for a given and fixed mean-field parameter, we solve the ergodic stochastic control problem faced by the representative player. In Section 4 we then prove the existence and uniqueness of the mean-field equilibrium, while in Section 5 we introduce and solve a MFG of optimal extraction. Finally, technical proofs are collected in Appendix A.

2. Problem Formulation

2.1. **Probabilistic setting.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which satisfies the usual conditions, on which it is defined a one-dimensional Brownian motion $\{W_t\}_{t\geq 0}$ and denote by $\mathbb{F} := \{\mathcal{F}_t^W\}_{t\geq 0}$ the filtration which is generated by W, as usual augmented by $\mathbb{P}$-null sets of $\mathcal{F}$. Let

$$\mathcal{A} := \{\ \{\xi_t\}_{t\geq 0},\ \mathbb{F}\text{-adapted, nondecreasing, left-continuous and such that } \xi_0 = 0,\ \text{a.s.}\ \}, \tag{2.1}$$

and set $\mathbb{R} := (-\infty, \infty)$ and $\mathbb{R}_+ := (0, \infty)$. Then, for given $\xi \in \mathcal{A}$ and Borel-measurable functions $b : \mathbb{R}_+ \to \mathbb{R}$, $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$, we introduce the $\mathbb{R}_+$-valued process X^ξ with dynamics under $\mathbb{P}$

$$dX_t^\xi = b(X_t^\xi)dt + \sigma(X_t^\xi)dW_t^{\mathbb{P}} - d\xi_t, \quad X_0^\xi = x \in \mathbb{R}_+. \tag{2.2}$$

The following assumption ensures, in particular, that there exists a unique strong solution to (2.2) for every $\xi \in \mathcal{A}$ and $x \in \mathbb{R}_+$ (see Theorem 7 Chapter V in [60]). In the following, we shall denote such a strong solution by $X^{x,\xi}$, when needed.

Assumption 2.1. *The following hold:*

(1) *The functions b and σ are continuously differentiable.*
(2) *There exists $C > 0$ and $\zeta > 0$, such that*

$$|b(x)| + |\sigma(x)| \leq C(1 + |x|^\zeta), \quad \textit{for any } x \in \mathbb{R}_+.$$

Denoting by X^0 the unique strong solution to (2.2) with $\xi \equiv 0$, Conditions (1) and (2) in Assumption 2.1 imply that X^0 is regular, meaning that, for any $x, y \in \mathbb{R}_+$, $\mathbb{P}_x(\beta_y < \infty) > 0$, where $\beta_y := \inf\{t \geq 0 : X_t^0 = y\}$, $\mathbb{P}_x$-a.s. For further details see Section II-1 in [10].

For arbitrary $x_0 > 0$, we define the scale function of X^0 under $\mathbb{P}$ as

$$S^{\mathbb{P}}(x) := \int_{x_0}^x S_x^{\mathbb{P}}(y)dy \quad \text{with} \quad S_x^{\mathbb{P}}(x) := \exp\left(- \int_{x_0}^x \frac{2b(x)}{\sigma^2(y)}dy \right), \quad x \in \mathbb{R}_+, \tag{2.3}$$

and the speed measure under $\mathbb{P}$

$$m^{\mathbb{P}}((x_0, x)) := \int_{x_0}^x \frac{2}{\sigma^2(y)S_x^{\mathbb{P}}(y)}dy, \quad x \in \mathbb{R}_+. \tag{2.4}$$

We make the following standing assumption, which ensures that 0 is not attainable for X^0.

Assumption 2.2. *The speed measure m under $\mathbb{P}$ of X^0 satisfies*

$$\lim_{x_0 \downarrow 0} m^{\mathbb{P}}((x_0, x)) < \infty, \quad \textit{for any } x \in \mathbb{R}_+. \tag{2.5}$$

Remark 2.1. *A mean-reverting uncontrolled process X^0 with $b(x) = \alpha(\kappa - x)$ and $\sigma(x) = \sigma x$, for positive κ, α and σ, satisfies Assumption 2.1 and 2.2.*

We also define $\mathcal{Q}$ to be the set of probability measures on $(\Omega, \mathcal{F})$, which are equivalent with respect to $\mathbb{P}$, i.e. $\mathcal{Q} := \{\mathbb{Q} \in \mathcal{P}(\Omega, \mathcal{F}) : \mathbb{Q} \sim \mathbb{P}\}$, where $\mathcal{P}(\Omega, \mathcal{F})$ is the set of probability measures on $(\Omega, \mathcal{F})$ endowed with the weak topology. In the rest of the paper, we adopt the following notation: $\mathbb{P}_x[\cdot] := \mathbb{P}[\cdot | X_0^\xi = x]$ and $\mathbb{E}_x^{\mathbb{P}}[\cdot] := \mathbb{E}^{\mathbb{P}}[\cdot | X_0^\xi = x]$. Also, for $\mathbb{Q} \in \mathcal{Q}$, we set $\mathbb{Q}_x[\cdot] := \mathbb{Q}[\cdot | X_0^\xi = x]$ and $\mathbb{E}_x^{\mathbb{Q}}[\cdot] := \mathbb{E}_x^{\mathbb{Q}}[\cdot | X_0^\xi = x]$.

2.2. **The ergodic mean-field game.** Within the previous probabilistic setting, we now introduce the ergodic mean-field game (ergodic MFG for short) which will be the main object of our study. We start with the definition of admissible controls.

Definition 2.1 (Admissible controls)**.** *Let $x \in \mathbb{R}_+$, $p > 1$ and $q := \frac{p}{p-1}$. We say that $\xi \in \mathcal{A}$ is admissible if $X_t^\xi > 0$ for any $t \geq 0$, $\mathbb{P}$-a.s.,*

$$\mathbb{E}_x^{\mathbb{P}}\big[\xi_T^p\big] + \mathbb{E}_x^{\mathbb{P}}\big[|X_T^\xi|^{2q}\big] + \mathbb{E}_x^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T |X_t^\xi|^4 dt\right)\right] < \infty, \quad \textit{for any } T < \infty, \tag{2.6}$$

and if

$$\limsup_{T\uparrow\infty}\frac{1}{T}\mathbb{E}_x^{\mathbb{P}}\big[|X_T^{\xi}|^3\big]=0. \tag{2.7}$$

We denote the set of admissible controls ξ as $\mathcal{A}_e(x)$.

The previous technical integrability conditions will be needed in the verification theorem (see Theorem 3.2 below), when proving optimality of a candidate value and candidate optimal control.

For a Borel-measurable function $g:\mathbb{R}\to\mathbb{R}$ (such that the subsequent quantities are well-posed), introduce the integral (cf. [46], [64], among others)

$$\int_0^t g(X_s^{\xi})\circ d\xi_s := \int_0^t g(X_s^{\xi})d\xi_s^c + \sum_{0\le s\le t}\int_0^{\Delta\xi_s} g(X_s^{\xi}-r)dr, \quad t\ge 0, \tag{2.8}$$

where ξ^c denotes the continuous part of ξ. Then, for any $(\xi,\mathbb{Q})\in\mathcal{A}_e(x)\times\mathcal{Q}$, the profit functional to be optimized is given by

$$J^{\epsilon}(x;\xi,\mathbb{Q},\theta):=\liminf_{T\to\infty}\frac{1}{T}\mathbb{E}_x^{\mathbb{Q}}\bigg[\int_0^T\pi(X_t^{\xi},\theta)dt-\int_0^T c(X_t^{\xi})\circ d\xi_t+\frac{1}{\epsilon}\log\bigg(\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T}\bigg)\bigg], \tag{2.9}$$

for $\epsilon>0$ and $\theta\in\mathbb{R}_+$. In (2.9), π is the instantaneous profit function and c the proportional cost function satisfying Assumption 2.3 below.

Assumption 2.3. *The function $\pi:\mathbb{R}_+^2\mapsto\mathbb{R}_+$ and $c:\mathbb{R}_+\to\mathbb{R}_+$ are such that:*

1. *$\pi(\cdot,\theta)\in C^2(\mathbb{R}_+)$, for any $\theta\in\mathbb{R}_+$;*
2. *$\pi(\cdot,\theta)$ is non-decreasing and concave, for any $\theta\in\mathbb{R}_+$;*
3. *$\pi_{x\theta}$ is continuous and it is such that $\pi_{x\theta}(x,\theta)<0$, for any $(x,\theta)\in\mathbb{R}_+^2$.*
4. *c is continuously differentiable, nonincreasing and bounded, i.e. there exists $0<\underline{c}\le\overline{c}<\infty$ such that $c(x)\in[\underline{c},\overline{c}]$ for any $x\in\mathbb{R}_+$. In particular, $\lim_{x\downarrow 0}c(x)=\overline{c}$ and $\lim_{x\uparrow\infty}c(x)=\underline{c}$.*

The parameter $\epsilon>0$ in (2.9) measures the level of ambiguity that the decision maker has towards the probabilistic model $\mathbb{Q}$ with respect to the reference probabilistic setting associated to $\mathbb{P}$. The following admissibility conditions clarify the structure of the Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t}$ in (2.9) (see also Definition 1 in [27]).

Definition 2.2 (Admissible measures)**.** *Let $q:=\frac{p}{p-1}$ for $p>1$ as in Definition 2.1. Given $x\in\mathbb{R}_+$, we say that $\mathbb{Q}\in\mathcal{Q}$ is admissible if*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(t):=\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t}=\exp\bigg(\int_0^t\psi(X_s^{\xi})dW_s-\frac{1}{2}\int_0^t\psi^2(X_s^{\xi})ds\bigg),\quad \xi\in\mathcal{A}_e(x),$$

where $\psi:\mathbb{R}\to\mathbb{R}$ is a bounded Borel-measurable locally Lipschitz function such that

$$\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(\frac{1}{2}\int_0^T\psi^2(X_s^{\xi})ds\bigg)\bigg]<\infty,\quad \xi\in\mathcal{A}_e(x),\ \textit{for any } T<\infty, \tag{2.10}$$

$$\mathbb{E}^{\mathbb{P}}\big[\psi^q(X_T^{\xi})\big]<\infty,\quad \xi\in\mathcal{A}_e(x),\quad \textit{for any } T<\infty, \tag{2.11}$$

the scale function of X^0 with respect to $\mathbb{Q}$, i.e.

$$S_x^{\mathbb{Q}}(x):=\exp\bigg(-\int_c^x\frac{2(b(y)+\psi(y)\sigma(y))}{\sigma^2(y)}dy\bigg),\quad x\in\mathbb{R}_+, \tag{2.12}$$

is well-defined.

In the following analysis, we shall refer to $\psi_t = \psi(X_t^{x,\xi}),\ \xi \in \mathcal{A}_e(x)$, as the Girsanov kernel of $\mathbb{Q}$, and will denote the set of admissible $\mathbb{Q}$ as $\widehat{\mathcal{Q}}(x) \subseteq \mathcal{Q}$. Let then $\mathcal{S}(x) := \mathcal{A}_e(x) \times \widehat{\mathcal{Q}}(x)$. By choosing an admissible $\mathbb{Q} \in \widehat{\mathcal{Q}}(x)$, and writing for simplicity $\psi_t := \psi(X_t^{x,\xi}),\ \xi \in \mathcal{A}_e(x),\ x \in \mathbb{R}_+$, we have

$$\begin{aligned}
\mathbb{E}_x^{\mathbb{Q}}\Big[\log\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}(T)\Big)\Big] &= \mathbb{E}_x^{\mathbb{Q}}\Big[\log\Big(\exp\Big(\int_0^T \psi_s dW_s - \frac{1}{2}\int_0^T \psi_s^2 ds\Big)\Big)\Big] \\
&= \mathbb{E}_x^{\mathbb{Q}}\Big[\int_0^T \psi_s dW_s - \frac{1}{2}\int_0^T \psi_s^2 ds\Big] \\
&= \mathbb{E}_x^{\mathbb{Q}}\Big[\Big(\int_0^T \psi_s \underbrace{(dW_s - \psi_s ds)}_{=dW^{\mathbb{Q}}} + \frac{1}{2}\int_0^T \psi_s^2 ds\Big)\Big] \\
&= \mathbb{E}_x^{\mathbb{Q}}\Big[\frac{1}{2}\int_0^T \psi_t^2 dt\Big],
\end{aligned}$$

so that for $(\xi, \mathbb{Q}) \in \mathcal{S}(x)$ the payoff functional (2.9) rewrites as

$$J^\epsilon(x;\xi,\mathbb{Q},\theta) := \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x^{\mathbb{Q}}\Big[\int_0^T \pi(X_t^\xi,\theta)dt - \int_0^T c(X_t^\xi)\circ d\xi_t + \frac{1}{2\epsilon}\int_0^T \psi_t^2 dt\Big]. \tag{2.13}$$

The parameter θ drives the interaction of the representative player with the population (see Definition 2.3 below). Its representation at equilibrium involves the functions $F : \mathbb{R}_+ \to \mathbb{R}_+$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfy the following requirements.

Assumption 2.4. *$F : \mathbb{R}_+ \to \mathbb{R}_+$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ are such that:*

1. *F and f are strictly increasing and continuously differentiable functions;*
2. *for $\delta \in (0,1)$, there exists $C > 0$ such that:*
   (a) $|f(x)| \le C(1+|x|^\delta), \quad |F(x)| \le C(1+|x|^{\frac{1}{\delta}})$,
   (b) $\big|F(x) - F(y)\big| \le C(1+|x|+|y|)^{\frac{1}{\delta}-1}|x-y|$;
3. $\lim_{x\uparrow\infty} F(x) = \lim_{x\uparrow\infty} f(x) = \infty$.

Remark 2.2. *As a benchmark example (see Remark 2.3 below) we may take*

$$\pi(x,\theta) = x^\delta(\theta^{-(1+\delta)} + \eta), \quad c(x) = c, \quad f(x) = x^\delta, \quad \textit{and,} \quad F(x) = x^{1/\delta} \tag{2.14}$$

for $\eta > 0$, $\delta \in (0,1)$, $c > 0$. In this case, Assumptions 2.3 and 2.4 hold.

The following definition finally introduces the notion of optimality for the considered MFG.

Definition 2.3 (Ergodic MFG Equilibrium)**.** *For $x \in \mathbb{R}_+$ and $\epsilon > 0$, a tuple $(\xi^*(\theta^*), \mathbb{Q}^*(\theta^*), \theta^*) \in \mathcal{S}(x) \times \mathbb{R}_+$ is said to be an* ***equilibrium of the ergodic MFG*** *for the initial condition $x \in \mathbb{R}_+$ if*

1. (a) *$J^\epsilon(x;\xi^*(\theta^*),\mathbb{Q}^*(\theta^*),\theta^*) \ge J^\epsilon(x;\xi,\mathbb{Q}^*(\theta^*),\theta^*)$, for any $\xi \in \mathcal{A}_e(x)$;*
   (b) *$J^\epsilon(x;\xi^*(\theta^*),\mathbb{Q}^*(\theta^*),\theta^*) \le J^\epsilon(x;\xi^*(\theta^*),\mathbb{Q},\theta^*)$, for any $\mathbb{Q} \in \widehat{\mathcal{Q}}(x)$.*
2. *The optimally controlled state process $X^{\xi^*(\theta^*)}$ under $\mathbb{Q}^*(\theta^*)$ admits a stationary distribution ν^{θ^*} and the consistency condition $\theta^* = F\big(\int_{\mathbb{R}_+} f(x)\nu^{\theta^*}(dx)\big)$ holds true.*

In order to solve the ergodic MFG we follow a three-step approach:

1. For $\theta \in \mathbb{R}_+$, we find $\xi^*(\theta)$ and $\mathbb{Q}^*(\theta)$ satisfying Definition 2.3-(1) (with θ^* replaced by θ).
2. We determine the stationary distribution ν^θ of $X^{\xi^*(\theta)}$ under $\mathbb{Q}^*(\theta)$.
3. We solve for the fixed-point problem deriving from the consistency condition as in Definition 2.3-(2).

Remark 2.3. *The benchmark example of Remark 2.2 relates to a MFG of optimal extraction (see Section 5 for more details), where: c is the cost associated to the extraction of a resource; π is the profit accrued from the production of a final good, produced having as input the extracted resource; η*

is a fixed price (in absence of competition); while θ represents the equilibrium stationary price arising from a competing market where symmetric producing companies face isoelastic demand functions à la Spence-Dixit-Stiglitz (cf. Section 3 in [1] and Section 2.1 in [13] for a micro-foundation).

3. Solving the Ergodic Zero-Sum Game

Here, for $\theta \in \mathbb{R}_+$ given and fixed, we consider the zero-sum game between a singular controller acting on $\xi \in \mathcal{A}_e(x)$ and an adverse player (Nature) choosing $\mathbb{Q} \in \widehat{\mathcal{Q}}(x)$. Then, the aim is to determine

$$\lambda^\epsilon(\theta) := \sup_{\xi \in \mathcal{A}_e(x)} \inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}(x)} J^\epsilon(x; \xi, \mathbb{Q}, \theta). \tag{3.1}$$

Notice that $\lambda^\epsilon(\theta)$ is expected to be independent of x given the ergodic setting. In order to tackle (3.1), we let $V^\epsilon(\cdot, \theta) : \mathbb{R}_+ \to \mathbb{R}$ and $\lambda^\epsilon(\theta) \in \mathbb{R}_+$ to be determined such that $V^\epsilon(\cdot, \theta) \in C^2(\mathbb{R}_+)$ and the pair $(V^\epsilon(\cdot, \theta), \lambda^\epsilon(\theta))$ solves the variational inequality

$$\max\left\{ \inf_{p \in \mathbb{R}} \{\mathcal{L}^{p,\epsilon} V^\epsilon(x, \theta) + \frac{1}{2\epsilon} p^2\} + \pi(x, \theta) - \lambda^\epsilon(\theta), -V^\epsilon_x(x, \theta) - c(x) \right\} = 0, \quad x \in \mathbb{R}_+, \tag{3.2}$$

where, for $p \in \mathbb{R}$ and $f \in C^2(\mathbb{R}_+)$,

$$\mathcal{L}^{p,\epsilon} f(x) := \frac{1}{2}\sigma^2(x) f_{xx}(x) + (b(x) + \sigma(x)p) f_x(x). \tag{3.3}$$

In the following, V^ϵ will be called potential function. It is clear that the infimum with respect to $p \in \mathbb{R}$ appearing in (3.2) is attained. Hence, we let

$$(\mathcal{L}^\epsilon f)(x) := \inf_{p \in \mathbb{R}} \{\mathcal{L}^{p,\epsilon} f(x) + \frac{1}{2\epsilon} p^2\} = \frac{1}{2}\sigma^2(x) f_{xx}(x) + b(x) f_x(x) - \frac{\epsilon}{2}\sigma^2(x)(f_x(x))^2, \tag{3.4}$$

and (3.2) rewrites as

$$\max\{\mathcal{L}^\epsilon V^\epsilon(x, \theta) + \pi(x, \theta) - \lambda^\epsilon(\theta), -V^\epsilon_x(x, \theta) - c(x)\} = 0, \quad x \in \mathbb{R}_+. \tag{3.5}$$

For the subsequent analysis, it is important to introduce the function $\ell^\epsilon : \mathbb{R}_+ \to \mathbb{R}$, defined as

$$\ell^\epsilon(x, \theta) := -b(x)c(x) + \pi(x, \theta) - \frac{1}{2}\sigma^2(x)(\epsilon c^2(x) + c_x(x)), \tag{3.6}$$

which satisfies the following assumption.

Assumption 3.1. *The following hold:*

(1) *For any $\epsilon > 0$, there exist $\widehat{x}_\epsilon(\theta) \in \mathbb{R}_+$ such that*

$$(\ell^\epsilon)_x(x, \theta) \begin{cases} > 0, & x < \widehat{x}_\epsilon(\theta) \\ = 0, & x = \widehat{x}_\epsilon(\theta) \\ < 0, & x > \widehat{x}_\epsilon(\theta). \end{cases} \tag{3.7}$$

(2) *One has that*

$$\lim_{x \uparrow \infty} \ell^\epsilon(x, \theta) = -\infty \quad \textit{and} \quad \ell^\epsilon(0, \theta) := \lim_{x \downarrow 0} \ell^\epsilon(x, \theta) \textit{ is finite}. \tag{3.8}$$

(3) *One has that $\widehat{\underline{x}}_\epsilon(\theta) := \inf\{x \geq \widehat{x}_\epsilon(\theta) : \ell^\epsilon(x, \theta) = \ell^\epsilon(0, \theta)\}$ is finite.*

Remark 3.1. *The process of Remark 2.1 and the setting of Remark 2.2 are such that Assumption 3.1 is satisfied.*

Remark 3.2. *Consider the Verhulst-Pearl logistic model with dynamics*

$$dX_t = X_t(\kappa - \alpha X_t)dt + \sigma X_t dW_t^{\mathbb{P}}, \quad X_0 = x, \tag{3.9}$$

where $\kappa, \alpha, \sigma > 0$. It is clear that Assumption 2.1 is satisfied and, moreover, one can check that Assumption 2.2 is valid. However, for the running profit π and proportional cost c as specified in Remark 2.2, it follows that Assumptions 3.7 holds only if condition $2\alpha - \epsilon\sigma^2 c < 0$ is satisfied.

3.1. **Existence of a solution to (3.5).** To prove existence and uniqueness of a classical solution to (3.5), we follow *the shooting method* as in [27] (see also Section 7.3 in [62]). To that end, for arbitrarily fixed $\beta \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}$, we introduce the following auxiliary boundary-value problem for $\phi_\beta^\gamma(\cdot, \theta) : \mathbb{R}_+ \to \mathbb{R}$:

(3.10)
$$\begin{cases} \frac{1}{2}\sigma^2(x)(\phi_\beta^\gamma)_x(x,\theta) + b(x)\phi_\beta^\gamma(x,\theta) - \frac{\epsilon}{2}\sigma^2(x)(\phi_\beta^\gamma)^2(x,\theta) = \ell^\epsilon(\beta,\theta) - \pi(x,\theta) + \gamma, \quad x < \beta, \\ \phi_\beta^\gamma(x,\theta) = -c(x), \quad x \geq \beta. \end{cases}$$

Proposition 3.1 (Regular solution to (3.10))**.** *For fixed $\beta \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}$, the boundary-value problem (3.10) has a unique solution $\phi_\beta^\gamma(\cdot, \theta) \in C^1(\mathbb{R}_+)$, for any $\theta \in \mathbb{R}_+$.*

Proof. To prove that problem (3.10) has a unique regular solution, we borrow the argument of Section 4.2 in [27]. We introduce the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x,\theta) := -\frac{\log(y(x,\theta))}{\epsilon}$, where $y(\cdot,\theta) : \mathbb{R}_+ \to \mathbb{R}_+$ solves

$$\begin{cases} \frac{1}{2}\sigma^2(x)y_{xx}(x,\theta) + b(x)y_x(x,\theta) + (\ell^\epsilon(\beta,\theta) - \pi(x,\theta) + \gamma)\epsilon y(x,\theta) = 0, \quad x \in (0,\beta) \\ y(x,\theta) = \frac{1}{c(x)}, \quad y_x(x,\theta) = \epsilon, \quad x \geq \beta. \end{cases} \tag{3.11}$$

Hence, $f_x(x)$ solves (3.10). In (3.11), all the coefficient are continuous due to the Assumptions (2.1) and (2.3). Hence, by Theorem 3.6.2 in [25], there exists a unique regular solution on $[\alpha, \infty)$, for every $\alpha \in (0, \beta)$. Since the Cole-Hopf transformation is one-to-one and onto we obtain existence and uniqueness of a solution as in the claim. ☐

Lemma 3.1 (Perturbation of (3.10))**.** *For any $\beta \in \mathbb{R}_+$, there exists $C := C(\beta)$ such that,*

$$\sup_{x\in(0,\beta]} \left|\phi_\beta^\gamma(x,\theta) - \phi_\beta^0(x,\theta)\right| \leq C|\gamma|, \quad \textit{for sufficient small } \gamma \in \mathbb{R}. \tag{3.12}$$

Proof. The proof follows arguments completely similar to those employed in the proof of Lemma 2 in [27] and it is therefore omitted. ☐

In the sequel, we write ϕ_β for ϕ_β^0, and recall that $\phi_\beta(\cdot,\theta) \in C^1(\mathbb{R}_+)$ by Proposition 3.1. To proceed, we then define

$$B_\epsilon(\theta) := \{\beta \in \mathbb{R}_+ | \phi_\beta(x,\theta) \geq -c(x), \ x \in (0,\beta]\}, \quad \beta_\epsilon(\theta) := \inf B_\epsilon(\theta), \tag{3.13}$$

and we have the following result on the structure of the problem's state space.

Proposition 3.2. *The following hold:*

(1) $(0, \widehat{x}_\epsilon(\theta)] \cap B_\epsilon(\theta) = \emptyset$;
(2) $\underline{\widehat{x}}_\epsilon(\theta) \in B^\epsilon(\theta)$;
(3) $\widehat{x}_\epsilon(\theta) \leq \beta_\epsilon(\theta) \leq \underline{\widehat{x}}_\epsilon(\theta)$.

Proof. **Proof of (1):** Fix $\beta \in (0, \widehat{x}_\epsilon(\theta)]$ and $\gamma > 0$. We define the function $F_\beta^\gamma(x,\theta) := \phi_\beta^\gamma(x,\theta) + c(x)$ and, since for $x = \beta$ we have $\phi_\beta^\gamma(\beta,\theta) = -c(\beta)$, it holds $F_\beta^\gamma(\beta,\theta) = 0$. We then plug $x = \beta$ in (3.10) and obtain that

$$\frac{1}{2}\sigma^2(\beta)(\phi_\beta^\gamma)_x(\beta,\theta) = \ell^\epsilon(\beta,\theta) - \big(b(\beta)\phi_\beta^\gamma(\beta,\theta) - \frac{1}{2}\sigma^2(\beta)(\phi_\beta^\gamma)^2(\beta,\theta) + \pi(\beta,\theta)\big) + \gamma;$$

equivalently,

$$\frac{1}{2}\sigma^2(\beta)\partial_x(\phi^\gamma_\beta(\cdot,\theta)+c(\cdot))(\beta)=\ell^\epsilon(\beta,\theta)\\ -\underbrace{\big(-b(\beta)c(\beta)+\pi(\beta,\theta)-\frac{1}{2}\sigma^2(\beta)\big(\epsilon c^2(\beta)+c_x(\beta)\big)\big)}_{=\ell^\epsilon(\beta,\theta)}+\gamma,$$

that is,

$$\frac{1}{2}\sigma^2(\beta)(F^\gamma_\beta)_x(\beta)=\ell^\epsilon(\beta,\theta)-\ell^\epsilon(\beta,\theta)+\gamma=\gamma>0.$$

This implies that $(F^\gamma_\beta)_x(\beta,\theta)>0$. Arguing by contradiction, we want to show that for any $x\in(0,\beta)$ we have that $F^\gamma_\beta(x,\theta)<0$, which, thanks to Lemma 3.1, in turn allows to conclude that $F_\beta(x,\theta)\le 0$ for $x\le\beta$; that is, $\phi_\beta(x,\theta)\le -c(x)$ for $x\le\beta$. Hence, we assume that there exists $x_0(\theta):=\max\{x\in(0,\beta):\phi^\gamma_\beta(x,\theta)=-c(x)\}$. Then,

$$\frac{1}{2}\sigma^2(x_0(\theta))(\phi^\gamma_\beta)_x(x_0(\theta))+b(x_0(\theta))\phi^\gamma_\beta(x_0(\theta))-\frac{\epsilon}{2}\sigma^2(x_0(\theta))(\phi^\gamma_\beta)^2(x_0(\theta))+\pi(x_0(\theta))\\ =\ell^\epsilon(\beta,\theta)+\gamma,$$

which, rearranging terms, yields

$$\frac{1}{2}\sigma^2(x_0(\theta))\partial_x(\phi^\gamma_\beta(\cdot,\theta)+c(\cdot))(x_0(\theta))=\ell^\epsilon(\beta,\theta)\\ -\underbrace{\big(-b(x_0(\theta))c(x_0(\theta))+\pi(x_0(\theta),\theta)-\frac{1}{2}\sigma^2(x_0)\big(\epsilon c^2(x_0(\theta))+c_x(x_0(\theta))\big)\big)}_{=\ell^\epsilon(x_0(\theta),\theta)}+\gamma.$$

Hence, thanks to (3.7) in Assumption 3.1,

$$\frac{1}{2}\sigma^2(x_0(\theta))(F^\gamma_\beta)_x(x_0(\theta),\theta)=\underbrace{\big(\ell^\epsilon(\beta,\theta)-\ell^\epsilon(x_0(\theta),\theta)\big)}_{>0}+\gamma>0,$$

so that $(F^\gamma_\beta)_x(x_0(\theta),\theta)>0$, which contradicts Lemma A.1.

Proof of (2): As in the previous step, we define $F^\gamma_{\underline{\widehat{x}}_\epsilon(\theta)}(x,\theta):=\phi^\gamma_{\underline{\widehat{x}}_\epsilon(\theta)}(x,\theta)+c(x)$. For $\gamma<0$, we plug $x=\underline{\widehat{x}}_\epsilon(\theta)$ into the (3.10). Then, using the fact that $\phi^\gamma_{\underline{\widehat{x}}_\epsilon}(\underline{\widehat{x}}_\epsilon(\theta),\theta)=-c(\underline{\widehat{x}}_\epsilon(\theta))$, which yields $F^\gamma_{\underline{\widehat{x}}_\epsilon(\theta)}(\underline{\widehat{x}}_\epsilon(\theta),\theta)=0$, following the same arguments of Step 1 above we obtain

$$\frac{1}{2}\sigma^2(\underline{\widehat{x}}_\epsilon(\theta))(F^\gamma_{\underline{\widehat{x}}_\epsilon(\theta)})_x(\underline{\widehat{x}}_\epsilon(\theta),\theta)=\gamma<0,$$

which implies that $(F^\gamma_{\underline{\widehat{x}}_\epsilon(\theta)})_x(\underline{\widehat{x}}_\epsilon(\theta))<0$. We want to show that for any $x\in(0,\underline{\widehat{x}}_\epsilon(\theta))$ we have $\phi^\gamma_{\underline{\widehat{x}}_\epsilon(\theta)}(x,\theta)>-c(x)$, so to conclude by Lemma 3.1 that $\phi_{\underline{\widehat{x}}_\epsilon(\theta)}(x,\theta)\ge -c(x),\ x\in(0,\underline{\widehat{x}}_\epsilon(\theta))$.

We argue again by contradiction and assume that there exists $x_1(\theta):=\max\{x\in(0,\underline{\widehat{x}}_\epsilon(\theta)):\phi^\gamma_{\underline{\widehat{x}}_\epsilon(\theta)}(x,\theta)=-c(x)\}=\max\{x\in(0,\underline{\widehat{x}}_\epsilon(\theta)):F^\gamma_{\underline{\widehat{x}}_\epsilon(\theta)}(x,\theta)=0\}$. Hence, feeding $x=x_1(\theta)$ into (3.10) we have

$$\frac{1}{2}\sigma^2(x_1(\theta))(\phi^\gamma_{\underline{\widehat{x}}_\epsilon(\theta)})_x(x_1(\theta))+b(x_1(\theta))\phi^\gamma_{\underline{\widehat{x}}_\epsilon(\theta)}(x_1(\theta))\\ -\frac{\epsilon}{2}\sigma^2(x_1(\theta))(\phi^\gamma_{\underline{\widehat{x}}_\epsilon(\theta)})^2(x_1(\theta))+\pi(x_1(\theta))=\ell^\epsilon(\underline{\widehat{x}}_\epsilon(\theta),\theta)+\gamma.$$

Rearranging the terms we find that

$$\frac{1}{2}\sigma^2(x_1(\theta))\partial_x(\phi^{\gamma}_{\widehat{\underline{x}}_\epsilon(\theta)}(\cdot,\theta)+c(\cdot))_x(x_1(\theta)) = \ell^\epsilon(\widehat{\underline{x}}_\epsilon(\theta),\theta)$$
$$-\underbrace{\Big(-b(x_1(\theta))c(x_1(\theta))+\pi(x_1(\theta),\theta)-\frac{1}{2}\sigma^2(x_1(\theta))\big(\epsilon c^2(x_1(\theta))+c_x(x_1(\theta))\big)\Big)}_{=\ell^\epsilon(x_1(\theta),\theta)}+\gamma;$$

that is,

$$\frac{1}{2}\sigma^2(x_1(\theta))(F^{\gamma}_{\widehat{\underline{x}}_\epsilon(\theta)})_x(x_1(\theta)) = \ell^\epsilon(\widehat{\underline{x}}_\epsilon(\theta),\theta)-\ell^\epsilon(x_1(\theta),\theta)+\gamma<0,$$

where the last inequality comes from Assumption 3.1-(3.7). Hence, $(F^{\gamma}_{\widehat{\underline{x}}_\epsilon(\theta)})_x(x_1(\theta)) < 0$, which contradicts Lemma A.1.

Proof of (3): This follows from the previous steps. □

We also have the following result on the boundedness of $\phi_{\beta_\epsilon(\theta)}$.

Proposition 3.3 (Uniform boundedness of $\phi_{\beta_\epsilon(\theta)}(x,\theta)$). *For any $\theta\in\mathbb{R}_+$, there exists $M(\theta)>0$ such that $\left|\phi_{\beta_\epsilon(\theta)}(x,\theta)\right|\le M(\theta)$, for any $x\in[0,\infty)$.*

Proof. Let arbitrarily fixed $\theta\in\mathbb{R}_+$. In the sequel, we denote for simplicity $\phi_{\beta_\epsilon(\theta)}(\cdot,\theta)$ by $\phi(\cdot,\theta)$. Given that $\phi_{\beta_\epsilon(\theta)}(x,\theta)=-c(x)$ for $x\in[\beta_\epsilon(\theta),\infty)$ and c is bounded, it suffices to consider $x\in(0,\beta_\epsilon(\theta)]$. We then have on $(0,\beta_\epsilon(\theta))$

$$\begin{aligned}
&\frac{d}{dx}\left(\phi(x,\theta)\exp\left(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(y)}{\sigma^2(y)}dy\right)\right)\\
&=\left(\phi_x(x,\theta)+\frac{2b(x)}{\sigma^2(x)}\phi(x,\theta)\right)\exp\left(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(y)}{\sigma^2(y)}dy\right)\\
&=\left(\frac{2}{\sigma^2(x)}\lambda^\epsilon(\theta)-\frac{2\pi(x,\theta)}{\sigma^2(x)}+\frac{2b(x)}{\sigma^2(x)}\phi(x,\theta)+\epsilon\phi^2(x,\theta)\right.\\
&\qquad\left.+\frac{2b(x)}{\sigma^2(x)}\phi(x,\theta)\right)\exp\left(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(y)}{\sigma^2(y)}dy\right)\\
&=\left(\frac{2}{\sigma^2(x)}\lambda^\epsilon(\theta)-\frac{2\pi(x,\theta)}{\sigma^2(x)}+\epsilon\phi^2(x,\theta)\right)\exp\left(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(y)}{\sigma^2(y)}dy\right),
\end{aligned}\tag{3.14}$$

where in second equation we have used (3.10) for $\beta=\beta_\epsilon(\theta)$ and $\gamma=0$. Then, for $x_0<\beta_\epsilon(\theta)$, we integrate (3.14) from x_0 to $\beta_\epsilon(\theta)$ and exploiting monotonicity of $x\mapsto\pi(x,\theta)$ (cf. Assumption 2.3-(2)) we obtain

$$\begin{aligned}
&\int_{x_0}^{\beta_\epsilon(\theta)}\frac{d}{dx}\left(\phi(x,\theta)\exp\left(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(y)}{\sigma^2(y)}dy\right)\right)\ge\\
&\qquad\big(\lambda^\epsilon(\theta)-\pi(\beta_\epsilon(\theta),\theta)\big)\int_{x_0}^{\beta_\epsilon(\theta)}\frac{2}{\sigma^2(x)}\exp\left(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(y)}{\sigma^2(y)}dy\right)dx.
\end{aligned}\tag{3.15}$$

By applying condition $\phi(\beta_\epsilon(\theta),\theta)=-c(\beta_\epsilon(\theta))<0$, we get

$$\begin{aligned}
&-c(\beta_\epsilon(\theta))-\phi(x_0,\theta)\exp\left(-\int_{x_0}^{\beta_\epsilon(\theta)}\frac{2b(y)}{\sigma^2(y)}dy\right)\ge\\
&\qquad\big(\lambda^\epsilon(\theta)-\pi(\beta_\epsilon(\theta),\theta)\big)\int_{x_0}^{\beta_\epsilon(\theta)}\frac{2}{\sigma^2(x)}\exp\left(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(y)}{\sigma^2(y)}dy\right)dx;
\end{aligned}\tag{3.16}$$

equivalently,

$$\begin{aligned}
\phi(x,\theta) &\le \exp\bigg(\int_{x_0}^{\beta_\epsilon(\theta)} \frac{2b(y)}{\sigma^2(y)}dy\bigg)\bigg(-c(\beta_\epsilon(\theta)) \\
&\quad + \big(\pi(\beta_\epsilon(\theta),\theta)-\lambda^\epsilon(\theta)\big)\int_{x_0}^{\beta_\epsilon(\theta)} \frac{2}{\sigma^2(x)}\exp\bigg(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(y)}{\sigma^2(y)}dy\bigg)dx\bigg) \\
&\le \big(|\lambda^\epsilon(\theta)|+\pi(\beta_\epsilon(\theta),\theta)\big)\exp\bigg(\int_{x_0}^{\beta_\epsilon(\theta)} \frac{2b(y)}{\sigma^2(y)}dy\bigg)\int_{x_0}^{\beta_\epsilon(\theta)} \frac{2}{\sigma^2(x)}\exp\bigg(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(y)}{\sigma^2(y)}dy\bigg)dx \\
&= \big(|\lambda^\epsilon(\theta)|+\pi(\beta_\epsilon(\theta),\theta)\big)\int_{x_0}^{\beta_\epsilon(\theta)} \frac{2}{\sigma^2(x)}\exp\bigg(\int_{x_0}^{x}\frac{2b(y)}{\sigma^2(y)}dy\bigg)dx \\
&\le \big(|\lambda^\epsilon(\theta)|+\pi(\beta_\epsilon(\theta),\theta)\big)m^{\mathbb{P}}((0,\beta_\epsilon(\theta))) =: M(\theta),
\end{aligned}$$

where we have used that $S_x^{\mathbb{P}}(x) = \exp\big(-\int_{x_0}^x \frac{2b(y)}{\sigma^2(y)}dy\big)$ and speed measure under $\mathbb{P}$, $m^{\mathbb{P}}(x,\alpha)$ (cf. (2.4)). Finally, $M(\theta)$ is finite in accordance with Assumption 2.2. $\square$

Now we are in the position to introduce the candidate optimal potential function V^ϵ as

$$V^\epsilon(x,\theta) := \begin{cases} -\int_x^{\beta_\epsilon(\theta)} \phi_{\beta_\epsilon(\theta)}(y,\theta)dy, & 0 < x < \beta_\epsilon(\theta), \\ -\int_{\beta_\epsilon(\theta)}^x c(y)dy, & x \ge \beta_\epsilon(\theta), \end{cases} \tag{3.17}$$

where $\beta_\epsilon(\theta)$ given by (3.13) is then such that $\beta_\epsilon(\theta) := \inf\{x \in \mathbb{R}_+ : V_x^\epsilon(x,\theta) = -c(x)\}$.

Theorem 3.1 (Existence and Uniqueness of solution to (3.5))**.** *Let $\beta_\epsilon(\theta)$ as in (3.13) and V^ϵ as in (3.17). Defining $\lambda^\epsilon(\theta) := \ell^\epsilon(\beta_\epsilon(\theta),\theta))$, the couple $(V^\epsilon(\cdot,\theta),\lambda^\epsilon(\theta))$, with $V^\epsilon(\cdot,\theta) \in C^2(\mathbb{R}_+)$, is the unique solution to (3.5).*

Proof. First of all, we show that $V^\epsilon(\cdot,\theta) \in C^2(\mathbb{R}_+)$. By definition of $V^\epsilon(\cdot,\theta)$ (cf. (3.17)) it is sufficient to show that $V^\epsilon_{xx}(\beta_\epsilon(\theta),\theta),\theta) = -c_x(\beta_\epsilon(\theta))$. To that end, plugging $x = \beta_\epsilon(\theta)$ in (3.10) (for $\beta = \beta_\epsilon(\theta)$) we obtain

$$\begin{aligned}
\frac{1}{2}\sigma^2(\beta_\epsilon(\theta))\partial_x\phi_{\beta_\epsilon(\theta)}(\beta_\epsilon(\theta),\theta) &= \ell^\epsilon(\beta_\epsilon(\theta),\theta) - \pi(\beta_\epsilon(\theta),\theta) - b(\beta_\epsilon(\theta))\phi_{\beta_\epsilon(\theta)}(\beta_\epsilon(\theta),\theta) \\
&\quad + \frac{\epsilon}{2}\sigma^2(\beta_\epsilon(\theta))(\phi_{\beta_\epsilon(\theta)})^2(\beta_\epsilon(\theta),\theta) \\
&= \ell^\epsilon(\beta_\epsilon(\theta),\theta) - \pi(\beta_\epsilon(\theta),\theta) + b(\beta_\epsilon(\theta))c(\beta_\epsilon(\theta)) \\
&\quad + \frac{\epsilon}{2}\sigma^2(\beta_\epsilon(\theta))c^2(\beta_\epsilon(\theta)) = \frac{1}{2}\sigma^2(\beta_\epsilon(\theta))\partial_x c(\beta_\epsilon(\theta)),
\end{aligned} \tag{3.18}$$

where the last equation follows from (3.6). Hence, given the strict positivity of σ we conclude the desired equation.

We now move on by showing that $(V^\epsilon(\cdot,\theta),\lambda^\epsilon(\theta))$ solve (3.5). For $x \ge \beta_\epsilon(\theta)$, $V^\epsilon(\cdot,\theta)$ as in (3.17) satisfies

$$\begin{aligned}
V_x^\epsilon(x,\theta) &= -c(x) \\
\mathcal{L}^\epsilon V^\epsilon(x,\theta) + \pi(x,\theta) &= -\frac{1}{2}\sigma^2(x)c_x(x) - b(x)c(x) - \frac{\epsilon}{2}\sigma^2(x)c^2(x) + \pi(x,\theta) \\
&= \ell^\epsilon(x,\theta) \le \ell^\epsilon(\beta_\epsilon(\theta),\theta)
\end{aligned}$$

where the last inequality comes from the Assumption 3.7 as $\beta_\epsilon(\theta) \ge \widehat{x}_\epsilon(\theta)$ (cf. Proposition 3.2-(3)).

On the other hand, for $x \in (0,\beta_\epsilon(\theta))$ it is clear that, $V_x^\epsilon(x,\theta) = \phi_{\beta_\epsilon(\theta)}(x,\theta)$. Thanks to Proposition 3.13 and Lemma A.3 we have that $\beta_\epsilon(\theta) \in B_\epsilon(\theta)$, hence $V_x^\epsilon(x,\theta) \ge -c(x)$. To conclude, the equation

$$\mathcal{L}^\epsilon V^\epsilon(x,\theta) + \pi(x,\theta) = \ell^\epsilon(\beta_\epsilon(\theta),\theta) = \lambda^\epsilon(\theta), \quad x \in (0,\beta_\epsilon(\theta)),$$

is satisfied since $\phi_{\beta_\epsilon(\theta)}(\cdot,\theta)$ solves (3.10). □

3.2. **Verification Result.** In order to derive the optimal control rule we introduce the following definition.

Definition 3.1 (Skorokhod Reflection Problem)**.** *Let $\mathcal{D}[0,\infty)$ be the space of càdlàg processes on $[0,\infty)$. Given $x\in\mathbb{R}_+$, $\mathbb{Q}\in\widehat{\mathcal{Q}}(x)$, $\beta\in\mathbb{R}_+$, and $\epsilon>0$, the process $(X^\xi,\xi)\in\mathcal{D}[0,\infty)\times\mathcal{A}_e(x)$ is said to be the solution of the Skorokhod reflection problem* **SP**$(x,\beta;\mathbb{Q},\psi)$ *for the $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$ if it satisfies the following properties:*

(1) $X^\xi_t = x+\int_0^t(b(X^\xi_s)+\sigma(X^\xi_s)\psi(X^\xi_s))ds+\int_0^t\sigma(X^\xi_s)dW^{\mathbb{Q}}_s-\xi_t,\quad \mathbb{Q}\otimes dt\text{-a.s.}$

(2) $X^\xi_t\in(0,\beta],\quad \mathbb{Q}\otimes dt\text{-a.s.}$

(3) $\int_0^T \mathbf{1}_{\{X^\xi_s<\beta\}}d\xi_s=0,\quad \mathbb{Q}\otimes dt\text{-a.s.}$

Proposition 3.4. *For any $\theta\in\mathbb{R}_+$, there exists $\xi^*(\theta)\in\mathcal{A}_e(x)$ such that $(X^{\xi^*(\theta)},\xi^*(\theta))$ is the unique solution to* **SP**$(x,\beta_\epsilon(\theta);\mathbb{Q},\psi)$ *with*

$$\xi^*_t(\theta)=\sup_{s\le t}\big(I(X^{\xi^*}_s(\theta))-\beta_\epsilon(\theta)\big)^+, \tag{3.19}$$

where $I(X)_t:=x+\int_0^t\big(b(X_s)+\sigma(X_s)\psi(X_s)\big)ds+\int_0^t\sigma(X_s)dW^{\mathbb{Q}}_s$.

Proof. Since the properties of Theorem 4.1 in [63] are satisfied (recall that ψ is bounded and locally-Lipschitz continuous as $\mathbb{Q}\in\widehat{\mathcal{Q}}(x)$; cf. Definition 2.2), we obtain that **SP**$(x,\beta;\mathbb{Q},\psi)$ has a unique solution $(X^{\overline{\xi}},\overline{\xi})$ and $X^{\overline{\xi}}$ is pathwise unique. Thanks to (3.19), it is standard to see that $(X^{\xi^*(\theta)},\xi^*(\theta))$ satisfies the properties of **SP**$(x,\beta;\mathbb{Q},\psi)$, and by uniqueness we conclude that $\overline{\xi}=\xi^*(\theta)$. Furthermore, under Assumption 2.2, the state space of the optimally controlled process $X^{\xi^*(\theta)}$ is $(0,\beta_\epsilon(\theta)]$, with 0 being not attainable and $\beta_\epsilon(\theta)$ being reflecting. It this then a standard result in the theory of one-dimensional diffusion that the process $X^{\xi^*(\theta)}$ cannot reach 0 in finite time with positive probability. Finally, since $X^{\xi^*(\theta)}_t\in(0,\beta_\epsilon(\theta)]$, $\mathbb{Q}\otimes dt$-a.s., the integrability conditions in Definition 2.1 are satisfied. Thus, $\xi^*(\theta)\in\mathcal{A}_e(x)$. □

We then have the following verification theorem.

Theorem 3.2 (Verification Theorem)**.** *For every $x\in\mathbb{R}_+$, let $\xi^*(\theta)$ solve* **SP**$(x,\beta_\epsilon(\theta);\mathbb{Q}^*,\psi^*)$*, where $\mathbb{Q}^*(\theta)\in\widehat{\mathcal{Q}}(x)$ is such that $\frac{d\mathbb{Q}^*}{d\mathbb{P}}\big|_{\mathcal{F}_t}:=\psi^*_t$, with $\psi^*_t:=-\epsilon\sigma(X^{\xi^*(\theta)}_t)V^\epsilon_x(X^{\xi^*(\theta)}_t,\theta)$, $\mathbb{Q}^*\otimes dt-$a.s. Then $(\xi^*(\theta),\mathbb{Q}^*(\theta))\in\mathcal{A}_e(x)\times\widehat{\mathcal{Q}}(x)$ realizes a saddle point in (3.1) and*

$$\lambda^\epsilon(\theta)=-b(\beta_\epsilon(\theta))c(\beta_\epsilon(\theta))+\pi(\beta_\epsilon(\theta),\theta)-\frac{1}{2}\sigma^2(\beta_\epsilon(\theta))\big(\epsilon c^2(\beta_\epsilon(\theta))+c_x(\beta_\epsilon(\theta))\big). \tag{3.20}$$

Proof. We split the proof into two steps.

Step 1: Let $T>0$, $x\in\mathbb{R}_+$ and $\epsilon>0$. Recall Theorem 3.1 and introduce a sequence of $\mathbb{F}-$stopping times $(\tau^*_n(\theta))_{n\in\mathbb{N}}$ such that $\tau^*_n(\theta):=\inf\{t\ge0:X^{\xi^*(\theta)}_t\notin[1/n,n]\}$. Then, fixing $\mathbb{Q}\in\widehat{\mathcal{Q}}(x)$ with Girsanov kernel ψ as in Definition 2.2 and applying Itô-Meyer's formula to $(V^\epsilon(X^{\xi^*(\theta)}_{T\wedge\tau^*_n(\theta)},\theta))_{T\ge0}$, we have under $\mathbb{Q}$ (cf. also (2.8)) (recalling that $\xi^*(\theta)$ may cause a jump only at $t = 0$)

$$\begin{aligned} V^\epsilon(X^{\xi^*(\theta)}_{T\wedge\tau^*_n(\theta)},\theta) =& V^\epsilon(x,\theta) + \int_0^{T\wedge\tau^*_n(\theta)} \Big(\frac{1}{2}\sigma^2(X^{\xi^*(\theta)}_s)V^\epsilon_{xx}(X^{\xi^*(\theta)}_s,\theta) + \big(b(X^{\xi^*(\theta)}_s) \\ &+ \sigma(X^{\xi^*(\theta)}_s)\psi_s\big)V^\epsilon_x(X^{\xi^*(\theta)}_s,\theta)\Big)ds + \int_0^{T\wedge\tau^*_n(\theta)} \sigma(X^{\xi^*(\theta)}_s)V^\epsilon_x(X^{\xi^*(\theta)}_s,\theta)dW^{\mathbb{Q}}_s \\ &- \int_0^{T\wedge\tau^*_n(\theta)} V^\epsilon_x(X^{\xi^*(\theta)}_s,\theta)d(\xi^*)^c_s(\theta) - \big(V^\epsilon(X^{\xi^*(\theta)}_{0+}) - V^\epsilon(X^{\xi^*(\theta)}_0)\big). \end{aligned} \tag{3.21}$$

Noting that by continuity of $\sigma(\cdot)V^\epsilon_x(\cdot,\theta)$ we have $\mathbb{E}^{\mathbb{Q}}_x\Big[\int_0^{T\wedge\tau^*_n(\theta)}\sigma(X^{\xi^*(\theta)}_s)V^\epsilon_x(X^{\xi^*(\theta)}_s,\theta)dW_s\Big] = 0,$ for any $T>0,\ n\in\mathbb{N}$, by taking expectations in (3.21) with respect to $\mathbb{Q}$ we obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}_x\big[V^\epsilon(X^{\xi^*(\theta)}_{T\wedge\tau^*_n(\theta)},\theta)\big] \geq& V^\epsilon(x,\theta) + \mathbb{E}^{\mathbb{Q}}_x\Big[\int_0^{T\wedge\tau^*_n(\theta)}\big(\lambda^\epsilon(\theta) - \pi(X^{\xi^*(\theta)}_s,\theta) - \frac{1}{2\epsilon}(\psi_s)^2\big)ds\Big] \\ &- \mathbb{E}^{\mathbb{Q}}_x\Big[\int_0^{T\wedge\tau^*_n(\theta)} V^\epsilon_x(X^{\xi^*(\theta)}_s,\theta)d(\xi^*)^c_s(\theta)\Big] \\ &- \mathbf{1}_{\{x>\beta_\epsilon(\theta)\}}\int_0^{x-\beta_\epsilon(\theta)} V^\epsilon_x(x-r,\theta)dr, \end{aligned} \tag{3.22}$$

where we have also used the fact that, for any admissible ψ, $V^\epsilon(\cdot,\theta)$ satisfies (cf. (3.5))

$$\begin{aligned} &\frac{1}{2}\sigma^2(X^{\xi^*(\theta)}_s)V^\epsilon_{xx}(X^{\xi^*(\theta)}_s,\theta) + \big(b(X^{\xi^*(\theta)}_s) + \sigma(X^{\xi^*(\theta)}_s)\psi_s\big)V^\epsilon_x(X^{\xi^*(\theta)}_s,\theta) \\ &\qquad \geq \lambda^\epsilon(\theta) - \pi(X^{\xi^*(\theta)}_s,\theta) - \frac{1}{2\epsilon}(\psi_s)^2, \quad \mathbb{Q}\otimes dt-\text{a.s.}, \end{aligned}$$

and

$$\big(V^\epsilon(X^{\xi^*(\theta)}_{0+}) - V^\epsilon(X^{\xi^*(\theta)}_0)\big) = \mathbf{1}_{\{x>\beta_\epsilon(\theta)\}}\int_0^{x-\beta_\epsilon(\theta)} V^\epsilon_x(x-r,\theta)dr. \tag{3.23}$$

Since, $\int_0^\infty \mathbf{1}_{(0,\beta_\epsilon(\theta))}(X^{\xi^*(\theta)}_s)d(\xi^*)_s(\theta) = 0$, and $V^\epsilon_x(X^{\xi^*(\theta)}_t,\theta) = -c(X^{\xi^*(\theta)}_t)$ when $X^{\xi^*(\theta)}_t \geq \beta_\epsilon(\theta)$ $\mathbb{Q}\otimes dt-$a.s., from (3.22) we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}_x\big[V^\epsilon(X^{\xi^*(\theta)}_{T\wedge\tau^*_n(\theta)},\theta)\big] \geq& V^\epsilon(x,\theta) + \mathbb{E}^{\mathbb{Q}}_x\Big[\int_0^{T\wedge\tau^*_n(\theta)}\big(\lambda^\epsilon(\theta) - \pi(X^{\xi^*(\theta)}_s,\theta) - \frac{1}{2\epsilon}(\psi_s)^2\big)ds\Big] \\ &+ \mathbb{E}^{\mathbb{Q}}_x\Big[\int_0^{T\wedge\tau^*_n(\theta)} c(X^{\xi^*(\theta)}_s)d(\xi^*)^c_s(\theta)\Big] + \mathbf{1}_{\{x>\beta_\epsilon(\theta)\}}\int_0^{x-\beta_\epsilon(\theta)} c(x-r)dr. \end{aligned} \tag{3.24}$$

Rearranging the terms and recalling (2.8) we find

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}_x\big[V^\epsilon(X^{\xi^*(\theta)}_{T\wedge\tau^*_n(\theta)},\theta)\big] + \mathbb{E}^{\mathbb{Q}}_x\Big[\int_0^{T\wedge\tau^*_n(\theta)}\big(\pi(X^{\xi^*(\theta)}_s,\theta) + \frac{1}{2\epsilon}(\psi_s)^2\big)\Big] \geq& V^\epsilon(x,\theta) + \lambda^\epsilon(\theta)\mathbb{E}^{\mathbb{Q}}_x\big[T\wedge\tau^*_n(\theta)\big] \\ &+ \mathbb{E}^{\mathbb{Q}}_x\Big[\int_0^{T\wedge\tau^*_n(\theta)} c(X^{\xi^*(\theta)}_s)\circ d(\xi^*)_s(\theta)\Big]. \end{aligned} \tag{3.25}$$

Given that $X^{\xi^*(\theta)}_t \in (0,\beta_\epsilon(\theta)];\mathbb{Q}\otimes dt-$a.s., passing to the limits as $n\uparrow\infty$ in (3.25), invoking monotone and dominated convergence theorems, we can exchange limits with expectations and obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}_x\big[V^\epsilon(X^{\xi^*(\theta)}_T,\theta)\big] + \mathbb{E}^{\mathbb{Q}}_x\Big[\int_0^T\big(\pi(X^{\xi^*(\theta)}_s,\theta) + \frac{1}{2\epsilon}(\psi_s)^2\big)\Big] \geq& V^\epsilon(x,\theta) + \lambda^\epsilon(\theta)T \\ &+ \mathbb{E}^{\mathbb{Q}}_x\Big[\int_0^T c(X^{\xi^*(\theta)}_s)\circ d(\xi^*)_s(\theta)\Big]. \end{aligned} \tag{3.26}$$

Rearranging terms, dividing by T and sending $T \uparrow \infty$ in (3.26) we obtain that

$$
\begin{aligned}
&\liminf_{T\uparrow\infty} \frac{1}{T}\mathbb{E}^{\mathbb{Q}}_x\Big[\int_0^T \big(\pi(X^{\xi^*(\theta)}_s,\theta) + \frac{1}{2\epsilon}(\psi_s)^2\big) - \int_0^T c(X^{\xi^*(\theta)}_s)\circ d(\xi^*)_s(\theta)\Big] \\
&\qquad \geq \lambda^\epsilon(\theta) - \limsup_{T\uparrow\infty}\frac{1}{T}\mathbb{E}^{\mathbb{Q}}_x\big[V^\epsilon(X^{\xi^*(\theta)}_T,\theta)\big]. \qquad (3.27)
\end{aligned}
$$

Hence, given again that $X^{\xi^*(\theta)}_t \in (0,\beta_\epsilon(\theta)],\ \mathbb{Q}\otimes dt-$a.s. and $V^\epsilon(\cdot,\theta)$ is continuous on $(0,\beta_\epsilon(\theta)]$, we find

$$
\liminf_{T\uparrow\infty} \frac{1}{T}\mathbb{E}^{\mathbb{Q}}_x\Big[\int_0^T \big(\pi(X^{\xi^*(\theta)}_s,\theta) + \frac{1}{2\epsilon}(\psi_s)^2\big) - \int_0^T c(X^{\xi^*(\theta)}_s)\circ d(\xi^*)_s(\theta)\Big] \geq \lambda^\epsilon(\theta); \qquad (3.28)
$$

that is,

$$
J^\epsilon(x;\xi^*(\theta),\mathbb{Q},\theta) \geq \lambda^\epsilon(\theta), \quad \text{for any } \mathbb{Q}\in\widehat{\mathcal{Q}}(x). \qquad (3.29)
$$

Hence,

$$
\sup_{\xi\in\mathcal{A}_e(x)}\ \inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}(x)} J^\epsilon(x;\xi,\mathbb{Q},\theta) \geq \lambda^\epsilon(\theta) \qquad (3.30)
$$

Step 2: Let now $\xi \in \mathcal{A}_e(x)$ and introduce the sequence of $\mathbb{F}-$stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n := \inf\{t \geq 0 : X^\xi_t \notin [1/n, n]\}$. Recalling the structure of the operator $\mathcal{L}^\epsilon$ in (3.4), we define a measure $\mathbb{Q}^*(\theta)\in\widehat{\mathcal{Q}}(x)$ with Girsanov kernel ψ^* given by

$$
\begin{aligned}
\psi^*_t :&= \arg\min_{\psi}\Big\{\frac{1}{2}\sigma^2(X^\xi_t)V^\epsilon_{xx}(X^\xi_t,\theta) + \big(b(X^\xi) + \sigma(X^\xi_t)\psi_t\big)V^\epsilon_x(X^\xi_t) + \frac{1}{2\epsilon}\psi^2_t\Big\} \\
&= \arg\min_{\psi}\Big\{\sigma(X^\xi_t)\psi_t V^\epsilon_x(X^\xi_t,\theta) + \frac{1}{2\epsilon}\psi^2_t\Big\} \\
&= -\epsilon\sigma(X^\xi_t)V^\epsilon_x(X^\xi_t,\theta), \quad \mathbb{P}\otimes dt\text{-a.s.}
\end{aligned}
$$

We claim that $\mathbb{Q}^*(\theta)$ is admissible according to Definition 2.2. First of all, given that $V^\epsilon_x = \phi_{\beta_\epsilon(\theta)}$ by (3.17), from Proposition 3.3 and Assumption 2.1 we obtain that the map $x \mapsto -\epsilon\sigma^2(x)V^\epsilon_x(x,\theta)$ is locally-Lipschitz. Consequently, $S^{\mathbb{Q}^*(\theta)}_x$ as in (2.12) satisfies Assumption 2.2. Furthermore, from Assumption 2.1-(2) and Proposition 3.3, there exists $K(\beta_\epsilon(\theta),\theta) > 0$ such that $|\psi^*_t| \leq K(\beta_\epsilon(\theta),\theta)(1 + |X^\xi_t|^2)$, $\mathbb{P}\otimes dt$-a.s. Hence, (2.10) holds because $\xi\in\mathcal{A}_e(x)$ and due to the sublinear growth of σ. Finally, (2.11) is satisfied, again since $\xi\in\mathcal{A}_e(x)$. Hence, $\mathbb{Q}^*(\theta)$ is admissible.

Using Itô-Meyer's formula to $(V^\epsilon(X^\xi_{T\wedge\tau_n},\theta))_{T\geq 0}$ and taking expectations under $\mathbb{Q}^*(\theta)$, we obtain (recall that, for any $\xi\in\mathcal{A}_e(x)$, it holds $\xi_t = \xi^c_t + \sum_{s\leq t}\Delta\xi_s$, where ξ^c is the continuous part of ξ)

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}^*(\theta)}_x\big[V^\epsilon(X^\xi_{T\wedge\tau_n},\theta)\big] &= V^\epsilon(x,\theta) + \mathbb{E}^{\mathbb{Q}^*(\theta)}_x\Big[\int_0^{T\wedge\tau_n}\Big(\frac{1}{2}\sigma^2(X^\xi_s)V^\epsilon_{xx}(X^\xi_s,\theta) + \big(b(X^\xi_s) \\
&\quad + \sigma(X^\xi_s)\psi^*_s\big)V^\epsilon_x(X^\xi_s,\theta)\Big)ds\Big] - \mathbb{E}^{\mathbb{Q}^*(\theta)}_x\Big[\int_0^{T\wedge\tau_n} V^\epsilon_x(X^\xi_s,\theta)d\xi^c_s\Big] \\
&\quad - \mathbb{E}^{\mathbb{Q}^*(\theta)}_x\Big[\sum_{s\leq T\wedge\tau_n}\big(V^\epsilon(X^\xi_{s+}) - V^\epsilon(X^\xi_s)\big)\Big]. \qquad (3.31)
\end{aligned}
$$

Since

$$
\sum_{s\leq T\wedge\tau_n}\big(V^\epsilon(X^\xi_{s+}) - V^\epsilon(X^\xi_s)\big) = \mathbf{1}_{\{\Delta\xi_s>0\}}\int_0^{\Delta\xi_s} V^\epsilon_x(X^\xi_s - r,\theta)dr, \quad \mathbb{Q}^*(\theta) - \text{a.s.}, \qquad (3.32)
$$

then, using (3.32) and the observation $\frac{1}{2\epsilon}(\psi_t^*)^2 = \frac{\epsilon}{2}\sigma^2(X_t^\xi)(V_x^\epsilon)^2(X_t^\xi,\theta)$, $\mathbb{Q}^* \otimes dt$−a.s., (3.31) gives

$$\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[V^\epsilon(X_{T\wedge\tau_n}^\xi,\theta)\big] = V^\epsilon(x,\theta) + \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\bigg[\int_0^{T\wedge\tau_n}\Big(\mathcal{L}^\epsilon V^\epsilon(X_t^\xi,\theta) - \frac{\epsilon}{2}\sigma^2(X_t^\xi)(V_x^\epsilon)^2(X_t^\xi,\theta)\Big)dt\bigg]$$
$$- \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\bigg[\int_0^{T\wedge\tau_n} V_x^\epsilon(X_s^\xi,\theta)\circ d\xi_s\bigg]. \tag{3.33}$$

From (3.5) we have

$$\mathcal{L}^\epsilon V^\epsilon(X_t^\xi,\theta) \le \lambda^\epsilon(\theta) - \pi(X_s^\xi,\theta), \quad \mathbb{Q}^*(\theta)\otimes dt - \text{a.s.} \tag{3.34}$$

Rearranging terms and using (3.34) in (3.33), one gets

$$\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\bigg[\int_0^{T\wedge\tau_n}\Big(\pi(X_t^\xi,\theta) + \frac{1}{2\epsilon}(\psi_t^*)^2\Big)dt\bigg] \le V^\epsilon(x,\theta) + \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\bigg[\int_0^{T\wedge\tau_n} c(X_s^\xi,\theta)\circ d\xi_s\bigg]$$
$$+ \lambda^\epsilon(\theta)\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[T\wedge\tau_n\big] - \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[V^\epsilon(X_{T\wedge\tau_n}^\xi,\theta)\big]$$
$$- \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\bigg[\int_0^{T\wedge\tau_n}\big(V_x^\epsilon(X_s^\xi,\theta) + c(X_t^\xi)\big)\circ d\xi_s\bigg]. \tag{3.35}$$

Using now that $V_x^\epsilon(X_t^\xi,\theta) \ge -c(X_t^\xi)$, $\mathbb{Q}^*(\theta)\otimes dt$-a.s., we obtain, for some $K>0$,

$$\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\bigg[\int_0^{T\wedge\tau_n}\Big(\pi(X_t^\xi,\theta) + \frac{1}{2\epsilon}(\psi_t^*)^2\Big)dt\bigg] \le V^\epsilon(x,\theta) + \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\bigg[\int_0^{T\wedge\tau_n} c(X_s^\xi,\theta)\circ d\xi_s\bigg]$$
$$+ \lambda^\epsilon(\theta)\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[T\wedge\tau_n\big] - \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[V^\epsilon(X_{T\wedge\tau_n}^\xi,\theta)\big]$$
$$\le V^\epsilon(x,\theta) + \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\bigg[\int_0^{T\wedge\tau_n} c(X_s^\xi,\theta)\circ d\xi_s\bigg] + \lambda^\epsilon(\theta)\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[T\wedge\tau_n\big]$$
$$\le K\big(\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[|X_{T\wedge\tau_n}^\xi|\big] + 1\big), \tag{3.36}$$

where in the last equality we have used, thanks to (3.17), Proposition 3.3, and boundedness of $c(\cdot)$, that

$$\big|V^\epsilon(x,\theta)\big| \le K(1+|x|),$$

for some $K>0$, possibly depending on θ. By standard SDE estimates (due to the local Lipschitz property of μ,σ and ψ^*) there exists $M>0$ such that

$$\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[\sup_{t\le T}|X_t^\xi|\big] \le M\big(1+|x|+\mathbb{E}_x^{\mathbb{Q}^*(\theta)}[\xi_T]\big) < \infty, \tag{3.37}$$

since $\xi\in\mathcal{A}_e(x)$. Then, letting $n\uparrow\infty$ and invoking monotone convergence theorem yields

$$\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\bigg[\int_0^{T}\Big(\pi(X_t^\xi,\theta) + \frac{1}{2\epsilon}(\psi_t^*)^2\Big)dt\bigg] \le V^\epsilon(x,\theta) + \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\bigg[\int_0^{T} c(X_s^\xi,\theta)\circ d\xi_s\bigg]$$
$$+ \lambda^\epsilon(\theta)T + K\big(\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[|X_T^\xi|\big]+1\big) \tag{3.38}$$

Hence, rearranging terms in (3.38), dividing by T and passing $T\uparrow\infty$ we have

$$\liminf_{T\uparrow\infty}\frac{1}{T}\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\bigg[\int_0^{T}\Big(\pi(X_t^\xi,\theta) + \frac{1}{2\epsilon}(\psi_t^*)^2\Big)dt - \int_0^T c(X_s^\xi,\theta)\circ d\xi_s\bigg] \le \lambda^\epsilon(\theta), \tag{3.39}$$

where in (3.39) we have used the property $\liminf_n(v_n+r_n)\le\liminf_n v_n + \limsup r_n$ and the fact that

$$\limsup_{T\uparrow\infty}\frac{1}{T}\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[|X_T^\xi|\big] = \limsup_{T\uparrow\infty}\frac{1}{T}\mathbb{E}_x^{\mathbb{P}}\big[|X_T^\xi|\psi_T^*\big] = 0, \tag{3.40}$$

given that $|\psi_T^*| \le K(\beta_\epsilon(\theta),\theta)(1+|X_t^\xi|^2),\ \mathbb{P}\otimes dt$-a.s. and $\xi \in \mathcal{A}_e(x)$. Because (3.39) holds for any $\xi \in \mathcal{A}_e(x)$, we find

$$\inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}(x)} \sup_{\xi\in\mathcal{A}_e(x)} J^\epsilon(x,\xi,\mathbb{Q};\theta) \le \lambda^\epsilon(\theta), \tag{3.41}$$

and, due to Step 1 we conclude that

$$\lambda^\epsilon(\theta) = \inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}(x)} \sup_{\xi\in\mathcal{A}_e(x)} J^\epsilon(x,\xi,\mathbb{Q};\theta) = \sup_{\xi\in\mathcal{A}_e(x)} \inf_{\mathbb{Q}\in\widehat{\mathcal{Q}}(x)} J^\epsilon(x,\xi,\mathbb{Q};\theta). \tag{3.42}$$

□

Remark 3.3. *As a byproduct of the verification theorem, we have obtained in (3.42) that the zero-sum game between the decision maker choosing ξ and Nature choosing $\mathbb{Q}$ has a value.*

4. Mean-Field Equilibrium

In the following, we prove existence and uniqueness of the mean-field equilibrium (cf. Definition 2.3) by an application of Schauder-Tychonof fixed-point theorem. Let $\mathcal{P}(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$ be the space of probability measures on $\mathbb{R}_+$ with the Borel σ-field, endowed with the weak topology.

4.1. **Continuity and boundedness of the free-boundary with respect to θ.**

In this subsection, we establish continuity and bounds of the map $\theta \mapsto \beta_\epsilon(\theta),\ \theta\in\mathbb{R}_+$. For our subsequent analysis, we introduce the following assumptions.

Assumptions 4.1.

(1) *There exists $\kappa:\mathbb{R}_+\to\mathbb{R}_+$ continuously differentiable such that*

$$\lim_{\theta\downarrow 0}\pi(x,\theta) = \infty \quad \textit{and} \quad \lim_{\theta\uparrow\infty}\pi(x,\theta) = \kappa(x). \tag{4.1}$$

and $\pi_x(x,\theta) \ge \kappa_x(x)$ for any $(x,\theta)\in\mathbb{R}_+^2$.

(2) *There exist $\delta\in(0,1)$ and $C>0$ such that,*

$$\big|\pi(x,\theta_2)-\pi(x,\theta_1)\big| \le C(1+|x|^\delta)|\theta_2-\theta_1|, \tag{4.2}$$

for any $\theta_1,\theta_2\in\mathbb{R}_+$ and $x\in\mathbb{R}_+$.

(3) *The function $\underline{\ell}^\epsilon:\mathbb{R}_+\to\mathbb{R}$, with $\underline{\ell}^\epsilon(x) := -b(x)c(x)+\kappa(x)-\frac{1}{2}\sigma^2(x)(\epsilon c^2(x)+c_x(x))$, satisfies the following:*

 (a) *For any $\epsilon>0$, there exist $\widehat{y}_\epsilon\in\mathbb{R}_+$ such that,*

$$(\underline{\ell}^\epsilon)_x(x) \begin{cases} >0, & x<\widehat{y}_\epsilon \\ =0, & x=\widehat{y}_\epsilon \\ <0, & x>\widehat{y}_\epsilon. \end{cases} \tag{4.3}$$

 (b) *One has that*

$$\lim_{x\uparrow\infty}\underline{\ell}^\epsilon(x) = -\infty \quad \textit{and} \quad \underline{\ell}^\epsilon(0) := \lim_{x\downarrow 0}\underline{\ell}^\epsilon(x) \quad \textit{is finite}. \tag{4.4}$$

 (c) *One has that $\underline{\widehat{y}}_\epsilon := \inf\{x\ge\widehat{y}_\epsilon(\theta) : \underline{\ell}^\epsilon(x) = \underline{\ell}^\epsilon(0)\}$ is finite.*

Remark 4.1. *An example of π and κ for which both Assumptions 2.3 and 4.1 are satisfied is $\pi(x,\theta) = x^\delta(\theta^{-(1+\delta)}+\eta)$ and $\kappa(x)=\eta x^\delta$, where $\delta\in(0,1)$ and $\eta>0$.*

Our first result is related to the monotonicity of the map $\theta\mapsto\beta_\epsilon(\theta),\ \theta\in\mathbb{R}_+$.

Lemma 4.1. *The map $\theta\mapsto\beta_\epsilon(\theta),\theta\in\mathbb{R}_+$, is nonincreasing.*

Proof. Let $\theta_1,\theta_2\in\mathbb{R}_+$ with $\theta_1<\theta_2$. From Proposition 3.13 we know that $\beta_\epsilon(\theta_1)$ is well-defined and we introduce $\phi_{\beta_\epsilon(\theta_1)}(\cdot,\theta_1)$ and $\phi_{\beta_\epsilon(\theta_1)}(\cdot,\theta_2)$, which are the unique classical solutions to (3.10) for $\gamma=0$, $\beta=\beta_\epsilon(\theta_1)$ and $\theta=\theta_1$ and $\theta=\theta_2$, respectively. Hence, from Proposition A.1-(1) we obtain that $\phi_{\beta_\epsilon(\theta_1)}(x,\theta_2)\ge\phi_{\beta_\epsilon(\theta_1)}(x,\theta_1)\ge -c(x)$ for any $x\in(0,\beta_\epsilon(\theta_1)]$, which implies that $\beta_\epsilon(\theta_1)\in B_\epsilon(\theta_2)$ (cf. (3.13)) and $\beta_\epsilon(\theta_2)=\inf B_\epsilon(\theta_2)\le\beta_\epsilon(\theta_1)$. □

Lemma 4.2. *There exists $\underline{\beta}_\epsilon \in \mathbb{R}_+$, such that $\beta_\epsilon(\theta) \geq \underline{\beta}_\epsilon$ for any $\theta \in \mathbb{R}_+$.*

Proof. Let $\gamma > 0, \theta \in \mathbb{R}_+$, and define the function $\psi^\gamma(x,\theta) := \phi_{\beta_\epsilon(\theta)}(x,\theta) - \underline{\phi}^\gamma(x)$, where $\phi_{\beta_\epsilon(\theta)}(\cdot,\theta)$ satisfies (3.10) for $\beta = \beta_\epsilon(\theta)$ and $\gamma = 0$, and $\underline{\phi}^\gamma$ satisfies

$$\begin{cases} \frac{1}{2}\sigma^2(x)\underline{\phi}^\gamma_x(x) + b(x)\underline{\phi}^\gamma(x) - \frac{\epsilon}{2}\sigma^2(x)(\underline{\phi}^\gamma)^2(x) = \underline{\ell}^\epsilon(\beta_\epsilon(\theta)) - \kappa(x) - \gamma, \quad x \in (0, \beta_\epsilon(\theta)) \\ \underline{\phi}^\gamma(x) = -c(x), \quad x \in [\beta_\epsilon(\theta), \infty). \end{cases} \tag{4.5}$$

Based on the proof of Proposition 3.1, we can show that the function $\underline{\phi}^\gamma$ uniquely solves (4.5) and it is such that $\underline{\phi}^\gamma \in C^1(\mathbb{R}_+)$. Hence, it follows that $\psi^\gamma(\cdot,\theta)$ is the unique continuously differentiable solution to

$$\begin{cases} \frac{1}{2}\sigma^2(x)(\psi^\gamma)_x(x,\theta) + b(x)\psi^\gamma(x,\theta) - \frac{\epsilon}{2}\sigma^2(x)(\psi^\gamma)^2(x,\theta) - \epsilon\sigma^2\underline{\phi}(x)\psi^\gamma(x,\theta) \\ \qquad\qquad = \big(\big(\pi(\beta_\epsilon(\theta),\theta) - \kappa(\beta_\epsilon(\theta))\big) - \big(\pi(x,\theta) - \kappa(x)\big)\big) + \gamma, \quad x \in (0, \beta_\epsilon(\theta)) \\ \psi^\gamma(x,\theta) = 0, \quad x \in [\beta_\epsilon(\theta), \infty). \end{cases} \tag{4.6}$$

Notice that $\kappa_x(x) \leq \pi_x(x,\theta)$ for any $\theta \in \mathbb{R}_+$ (cf. Assumptions 2.3-(3) and Assumption 4.1-(1)), and that, thanks to Assumption 4.1-(1),

$$\big(\pi(\beta_\epsilon(\theta),\theta) - \kappa(\beta_\epsilon(\theta))\big) - \big(\pi(x,\theta) - \kappa(x)\big) = \int_x^{\beta_\epsilon(\theta)} \big(\pi(y,\theta) - \kappa(y)\big)_x dy \geq 0, \quad x \leq \beta_\epsilon(\theta).$$

Then, plugging $x = \beta_\epsilon(\theta)$ in (4.6), we obtain $\psi^\gamma_x(\beta_\epsilon(\theta),\theta) > 0$. We want to show that $\psi^\gamma(x,\theta) \leq 0$ for any $x \in (0, \beta_\epsilon(\theta)]$. Arguing by contradiction, we assume that there exists $z_0(\theta) := \sup\{x \in (0,\beta_\epsilon(\theta)) : \psi^\gamma(x,\theta) = 0\}$ (which is well-defined due to the continuity of $\psi^\gamma(\cdot,\theta)$; cf. Proposition 3.1, we plug $x = z_0(\theta)$ in (4.6)), and we obtain

$$\frac{1}{2}\sigma^2(z_0(\theta))(\psi^\gamma)_x(z_0(\theta),\theta) = \int_{z_0(\theta)}^{\beta_\epsilon(\theta)} \big(\pi(y,\theta) - \kappa(y)\big)_x dy + \gamma > 0, \tag{4.7}$$

where the last inequality follows from Assumptions 2.3-(2). Hence, we reach to a contradiction with Lemma A.1. Consequently, we have that $-c(x) \leq \phi_{\beta_\epsilon(\theta)}(x,\theta) \leq \underline{\phi}^\gamma(x)$ for any $x \in (0, \beta_\epsilon(\theta)]$. Finally, thanks to (4.1), (4.3), and (4.4), we can mimic the steps of the proof of Proposition 3.2 to show the existence of $\underline{\beta}_\epsilon := \sup\{x \in (0,\beta_\epsilon(\theta)] : \underline{\phi}^\gamma(x) \geq -c(x)\} > 0$, with $\underline{\beta}_\epsilon \leq \beta_\epsilon(\theta)$. Then we conclude as in Lemma 4.1. ☐

Lemma 4.3. *For any $\theta_1, \theta_2 \in \mathbb{R}_+$, there exists $C_0 := C_0(\theta_1,\theta_2) > 0$ such that*

$$\big|\lambda^\epsilon(\theta_2) - \lambda^\epsilon(\theta_1)\big| \leq C_0|\theta_2 - \theta_1|. \tag{4.8}$$

Proof. For arbitrary $\theta_1, \theta_2 \in \mathbb{R}_+$, we recall (2.13) and for convenience we denote $\mathbb{Q}^*_i := \mathbb{Q}^*(\theta_i),\ i = 1, 2$. Then

$$\begin{aligned} \lambda^\epsilon(\theta_2) - \lambda^\epsilon(\theta_1) &= \sup_{\xi \in \mathcal{A}_e(x)} \inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}(x)} J^\epsilon(x;\xi,\mathbb{Q},\theta_2) - \sup_{\xi \in \mathcal{A}_e(x)} \inf_{\mathbb{Q} \in \widehat{\mathcal{Q}}(x)} J^\epsilon(x;\xi,\mathbb{Q},\theta_1) \\ &\leq J^\epsilon(x;\xi^*(\theta_2),\mathbb{Q}^*_1,\theta_2) - J^\epsilon(x;\xi^*(\theta_2),\mathbb{Q}^*_1,\theta_1) \\ &\leq \limsup_{T\uparrow\infty} \frac{1}{T}\mathbb{E}^{\mathbb{Q}^*_1}_x\left[\int_0^T \big|\pi(X^{\xi^*(\theta_2)}_t,\theta_2) - \pi(X^{\xi^*(\theta_2)}_t,\theta_1)\big|dt\right] \\ &\leq C \limsup_{T\uparrow\infty} \frac{1}{T}\mathbb{E}^{\mathbb{Q}^*_1}_x\left[\int_0^T (1 + |X^{\xi^*(\theta_2)}_t|^\delta)dt\right]\big|\theta_2 - \theta_1\big|, \end{aligned}$$

where in the second inequality we have used the property $\liminf_n \alpha_n - \liminf_n \beta_n \leq \limsup_n(\alpha_n - \beta_n)$, and in the third inequality Assumption 4.1-(2). Since, $X^{\xi^*(\theta_2)}_t \in (0, \beta_\epsilon(\theta_2)],\ \mathbb{Q}^*_1$–a.s., we obtain that

$$\mathbb{E}^{\mathbb{Q}^*_1}_x\big[\big|X^{\xi^*(\theta_2)}_t\big|^\delta\big] \leq \beta^\delta_\epsilon(\theta_2) < \infty. \tag{4.9}$$

Hence,

$$\limsup_{T\uparrow\infty}\frac{1}{T}\mathbb{E}^{\mathbb{Q}_1^*}\bigg[\int_0^T(1+|X_t^{\xi^*(\theta_2)}|^\delta)dt\bigg]\leq\limsup_{T\uparrow\infty}\frac{1}{T}\int_0^T(1+\beta_\epsilon^\delta(\theta_2))dt=1+\beta_\epsilon^\delta(\theta_2)<\infty. \tag{4.10}$$

and we conclude. □

Our next result is about the continuity of the map $\theta\mapsto V_x^\epsilon(x,\theta),\ x\in\mathbb{R}_+$.

Proposition 4.1. *The map $\theta\mapsto V_x^\epsilon(x,\theta),\ x\in\mathbb{R}_+$, is locally Lipschitz continuous.*

Proof. Let $\theta_1,\theta_2\in\mathbb{R}_+$. We focus on the case of $\theta_1\leq\theta_2$, since the proof is analogous in the other case. By Lemma 4.1, we have that $\beta_\epsilon(\theta_2)\leq\beta_\epsilon(\theta_1)$. For $x\in\mathbb{R}_+$, we know that $V_x^\epsilon(\cdot,\theta_i)$ satisfies (3.10), for $\beta=\beta_\epsilon(\theta_i),\ i=1,2$ and $\gamma=0$. Hence,

$$\begin{aligned}V_x^\epsilon(x,\theta_1)-V_x^\epsilon(x,\theta_2)&=\big(V_x^\epsilon(x,\theta_1)-V_x^\epsilon(x,\theta_2)\big)\mathbf{1}_{\{x\in(0,\beta_\epsilon(\theta_2))\}}\\&\quad+\big(V_x^\epsilon(x,\theta_1)-V_x^\epsilon(x,\theta_2)\big)\mathbf{1}_{\{x\in[\beta_\epsilon(\theta_2),\beta_\epsilon(\theta_1)]\}}+\big(V_x^\epsilon(x,\theta_1)-V_x^\epsilon(x,\theta_2)\big)\mathbf{1}_{\{x\in[\beta_\epsilon(\theta_1),\infty)\}}\\&=\big(V_x^\epsilon(x,\theta_1)-V_x^\epsilon(x,\theta_2)\big)\mathbf{1}_{\{x\in(0,\beta_\epsilon(\theta_2))\}}+\big(V_x^\epsilon(x,\theta_1)+c(x)\big)\mathbf{1}_{\{x\in[\beta_\epsilon(\theta_2),\beta_\epsilon(\theta_1))\}},\end{aligned}\tag{4.11}$$

where we have used the fact that $V_x^\epsilon(x,\theta_i)=-c(x),\ x\in[\beta_\epsilon(\theta_1),\infty)$ for $i=1,2$.

Take now $x\in[\beta_\epsilon(\theta_2),\beta_\epsilon(\theta_1))$ and define $F_2^{(\theta_1,\theta_2)}(x):=V_x^\epsilon(x,\theta_1)+c(x)$. Notice that, actually, by Proposition 3.3, there exists $\overline{M}(\theta_1)>0$ such that $\sup_{x\geq0}\big|F_2^{(\theta_1,\theta_2)}(x)\big|\leq\overline{M}(\theta_1)$. We start by showing that there exists $C_2(\theta_1,\theta_2)>0$ such that $\big|F_2^{(\theta_1,\theta_2)}(x)\big|\leq C_2(\theta_1,\theta_2)|\theta_1-\theta_2|,\ x\in[\beta_\epsilon(\theta_2),\beta_\epsilon(\theta_1))$. To that end, notice that by (3.10) (for $\beta=\beta_\epsilon(\theta_1)$) $F_2^{(\theta_1,\theta_2)}$ satisfies

$$\begin{aligned}&\frac{1}{2}\sigma^2(x)\partial_xF_2^{(\theta_1,\theta_2)}+b(x)F_2^{(\theta_1,\theta_2)}-\frac{\epsilon}{2}\sigma^2(x)(F_2^{(\theta_1,\theta_2)})^2(x)+\epsilon\sigma^2(x)c(x)F_2^{(\theta_1,\theta_2)}(x)\\&\qquad=\big(\lambda^\epsilon(\theta_1)-\ell^\epsilon(x,\theta_2)\big)-\big(\pi(x,\theta_1)-\pi(x,\theta_2)\big),\ x\in[\beta_\epsilon(\theta_2),\beta_\epsilon(\theta_1)),\end{aligned}\tag{4.12}$$

with $F_2^{(\theta_1,\theta_2)}(\beta_\epsilon(\theta_1))=0$. For $x\in[\beta_\epsilon(\theta_2),\beta_\epsilon(\theta_1))$, by Assumption 3.1-(3.7) we have that

$$\ell^\epsilon(x,\theta_2)-\lambda^\epsilon(\theta_1)\leq\ell^\epsilon(\beta_\epsilon(\theta_2),\theta_2)-\lambda^\epsilon(\theta_1)=\lambda^\epsilon(\theta_2)-\lambda^\epsilon(\theta_1),\tag{4.13}$$

and, by the fundamental theorem of calculus (for $x\in[\beta_\epsilon(\theta_2),\beta_\epsilon(\theta_1))$, using (4.12) and (4.13) we have

$$\begin{aligned}0\leq F_2^{(\theta_1,\theta_2)}(x)&=F_2^{(\theta_1,\theta_2)}(\beta_\epsilon(\theta_1)))-\int_x^{\beta_\epsilon(\theta_1)}\partial_xF_2^{(\theta_1,\theta_2)}(y)dy\\&=-\int_x^{\beta_\epsilon(\theta_1)}\frac{2}{\sigma^2(y)}\bigg(-b(x)+\frac{\epsilon}{2}\sigma^2(y)F_2^{(\theta_1,\theta_2)}(y)-\epsilon\sigma^2(y)c(y)\bigg)F_2^{(\theta_1,\theta_2)}(y)dy\\&\quad-\int_x^{\beta_\epsilon(\theta_1)}\frac{2}{\sigma^2(y)}\bigg(\big(\lambda^\epsilon(\theta_1)-\ell^\epsilon(x,\theta_2)\big)-\big(\pi(y,\theta_1)-\pi(y,\theta_2)\big)\bigg)dy\\&\leq\int_x^{\beta_\epsilon(\theta_1)}\frac{2}{\sigma^2(y)}\bigg(+b(x)+\epsilon\sigma^2(y)c(y)\bigg)F_2^{(\theta_1,\theta_2)}(y)dy\\&\quad+\int_x^{\beta_\epsilon(\theta_1)}\frac{2}{\sigma^2(y)}\bigg(\big(\lambda^\epsilon(\theta_2)-\lambda^\epsilon(\theta_1)\big)-\big(\pi(y,\theta_2)-\pi(y,\theta_1)\big)\bigg)dy.\end{aligned}\tag{4.14}$$

Hence,

$$\begin{aligned}0\leq F_2^{(\theta_1,\theta_2)}(x)&\leq\bigg(\int_x^{\beta_\epsilon(\theta_1)}\frac{2}{\sigma^2(y)}dy\bigg)\big|\lambda^\epsilon(\theta_2)-\lambda^\epsilon(\theta_1)\big|+\int_x^{\beta_\epsilon(\theta_1)}\frac{2}{\sigma^2(y)}\big|\pi(y,\theta_2)-\pi(y,\theta_1)\big|dy\\&\quad+\int_x^{\beta_\epsilon(\theta_1)}\bigg(\frac{2|b(x)|}{\sigma^2(y)}+2\epsilon\overline{c}\bigg)F_2^{(\theta_1,\theta_2)}(y)dy.\end{aligned}\tag{4.15}$$

Then, by Grönwall inequality,

$$0 \le F_2^{(\theta_1,\theta_2)}(x) \le \Big[\Big(\int_x^{\beta_\epsilon(\theta_1)} \frac{2}{\sigma^2(y)}dy\Big)\big|\lambda^\epsilon(\theta_2)-\lambda^\epsilon(\theta_1)\big| + \int_x^{\beta_\epsilon(\theta_1)} \frac{2}{\sigma^2(y)}\big|\pi(x,\theta_1)-\pi(x,\theta_2)\big|dy\Big]$$
$$\cdot \exp\Big(\int_x^{\beta_\epsilon(\theta_1)} \Big(\frac{2|b(x)|}{\sigma^2(y)} + 2\epsilon\bar{c}\Big)dy\Big),$$

which, using Assumption 2.3 and Lemma 4.3, yields

$$\begin{aligned}
0 &\le F_2^{(\theta_1,\theta_2)}(x)\\
&\le \Big(C_0\Big(\int_x^{\beta_\epsilon(\theta_1)} \frac{2}{\sigma^2(y)}dy\Big) + \int_x^{\beta_\epsilon(\theta_1)} \frac{2C(1+|y|^\delta)}{\sigma^2(y)}dy\Big)\\
&\quad\cdot \exp\Big(\int_x^{\beta_\epsilon(\theta_1)} \Big(\frac{2|b(x)|}{\sigma^2(y)} + 2\epsilon\bar{c}\Big)dy\Big)|\theta_2-\theta_1|\\
&\le \Big(C_0\Big(\int_{\beta_\epsilon(\theta_2)}^{\beta_\epsilon(\theta_1)} \frac{2}{\sigma^2(y)}dy\Big) + \int_{\beta_\epsilon(\theta_2)}^{\beta_\epsilon(\theta_1)} \frac{2C(1+|y|^\delta)}{\sigma^2(y)}dy\Big)\\
&\quad\cdot \exp\Big(\int_{\beta_\epsilon(\theta_2)}^{\beta_\epsilon(\theta_1)} \Big(\frac{2|b(x)|}{\sigma^2(y)} + 2\epsilon\bar{c}\Big)dy\Big)|\theta_2-\theta_1| =: C_2(\theta_1,\theta_2)|\theta_2-\theta_1|.
\end{aligned}\tag{4.16}$$

This gives the desired result.

Take now $x \in (0, \beta_\epsilon(\theta_2)]$. We want to show that there exists $C_1(x,\theta_1,\theta_2) > 0$ such that, $F_1^{(\theta_1,\theta_2)}(x) := V_x^\epsilon(x,\theta_1) - V_x^\epsilon(x,\theta_2)$, satisfies $\big|F_1^{(\theta_1,\theta_2)}(x)\big| \le C_1(x,\theta_1,\theta_2)|\theta_2-\theta_1|,\ x \in (0,\beta_\epsilon(\theta_2)]$. Notice that $F_1^{(\theta_1,\theta_2)}$ is the unique classical solution to

$$\begin{aligned}
&\frac{1}{2}\sigma^2(x)\partial_x F_1^{(\theta_1,\theta_2)}(x) + b(x)F_1^{(\theta_1,\theta_2)}(x) - \frac{\epsilon}{2}\sigma^2(x)(F_1^{(\theta_1,\theta_2)})^2(x) - \epsilon\sigma^2(x)V_x^\epsilon(x,\theta_1)F_1^{(\theta_1,\theta_2)}(x)\\
&= \big(\lambda^\epsilon(\theta_1)-\lambda^\epsilon(\theta_2)\big) - \big(\pi(x,\theta_1)-\pi(x,\theta_2)\big),\ x \in (0,\beta_\epsilon(\theta_2)),
\end{aligned}\tag{4.17}$$

with $F_1^{(\theta_1,\theta_2)}(\beta_\epsilon(\theta_2)) = F_2^{(\theta_1,\theta_2)}(\beta_\epsilon(\theta_2))$. Following the same steps as in the case of F_2, we have that

$$\begin{aligned}
&\big|F_1^{(\theta_1,\theta_2)}(x)\big| = \big|F_1^{(\theta_1,\theta_2)}(\beta_\epsilon(\theta_2)) - \int_x^{\beta_\epsilon(\theta_2)} \partial_x F_1^{(\theta_1,\theta_2)}(y)dy\big|\\
&\le \big|F_1^{(\theta_1,\theta_2)}(\beta_\epsilon(\theta_2))\big| + \int_x^{\beta_\epsilon(\theta_2)} \frac{2}{\sigma^2(y)}\big| - b(x) + \frac{\epsilon}{2}\sigma^2(y)F_1^{(\theta_1,\theta_2)}(y) + \epsilon\sigma^2(y)V_x^\epsilon(y,\theta_1)\big| \cdot \big|F_1^{(\theta_1,\theta_2)}(y)\big|dy\\
&\quad + \int_x^{\beta_\epsilon(\theta_2)} \frac{2}{\sigma^2(y)}\Big(\big|\lambda^\epsilon(\theta_2)-\lambda^\epsilon(\theta_1)\big| + \big|\pi(y,\theta_2)-\pi(y,\theta_1)\big|\Big)dy\\
&\le \big|F_1^{(\theta_1,\theta_2)}(\beta_\epsilon(\theta_2))\big| + \int_x^{\beta_\epsilon(\theta_2)} \Big(\frac{2|b(x)|}{\sigma^2(y)} + \epsilon\big|F_1^{(\theta_1,\theta_2)}(y)\big| + 2\epsilon\big|V_x^\epsilon(y,\theta_1)\big|\Big)\big|F_1^{(\theta_1,\theta_2)}(y)\big|dy\\
&\quad + \Big(\int_x^{\beta_\epsilon(\theta_2)} \frac{2}{\sigma^2(y)}dy\Big)\big|\lambda^\epsilon(\theta_2)-\lambda^\epsilon(\theta_1)\big| + \int_x^{\beta_\epsilon(\theta_2)} \frac{2}{\sigma^2(y)}\big|\pi(y,\theta_2)-\pi(y,\theta_1)\big|dy.
\end{aligned}\tag{4.18}$$

From (4.16), we know that $\big|F_2^{(\theta_1,\theta_2)}(x)\big| \le C_2(\theta_1,\theta_2)|\theta_2-\theta_1|$ on $[\beta_\epsilon(\theta_2),\beta_\epsilon(\theta_1))$. Moreover, by Proposition 3.3, there exist $\widehat{M}(\theta_1,\theta_2) > 0$ and $M(\theta_1) > 0$ such that $\big|F_1^{(\theta_1,\theta_2)}(x)\big| \le \widehat{M}(\theta_1,\theta_2)$ and $\big|V_x^\epsilon(x,\theta_1)\big| \le M(\theta_1)$, for any $x \in [0,\infty)$. These facts, combined with Assumption 4.1-(2) and

Lemma 4.3, gives

$$\begin{aligned}\big|F_1^{(\theta_1,\theta_2)}(x)\big| &\leq \bigg(C_2(\theta_1,\theta_2)+C_0\int_x^{\beta_\epsilon(\theta_2)}\frac{2}{\sigma^2(y)}dy + C\int_x^{\beta_\epsilon(\theta_2)}\frac{2(1+|y|^\delta)}{\sigma^2(y)}dy\bigg)\big|\theta_2-\theta_1\big| \\ &+\int_x^{\beta_\epsilon(\theta_2)}\bigg(\frac{2|b(x)|}{\sigma^2(y)}+\epsilon\widehat{M}(\theta_1,\theta_2)+2\epsilon M(\theta_1)\big)\bigg)\big|F_1^{(\theta_1,\theta_2)}(y)\big|dy \end{aligned}\tag{4.19}$$

Then, by Grönwall inequality, for suitable $K(\theta_1,\theta_2)>0$,

$$\begin{aligned}\big|F_1^{(\theta_1,\theta_2)}(x)\big| &\leq \bigg[C_1(\theta_1,\theta)+C_0\int_x^{\beta_\epsilon(\theta_2)}\frac{2}{\sigma^2(y)}dy + C\int_x^{\beta_\epsilon(\theta_2)}\frac{2(1+|y|^\delta)}{\sigma^2(y)}dy\bigg] \\ &\cdot\exp\bigg(\int_x^{\beta_\epsilon(\theta_2)}\bigg(\frac{2|b(x)|}{\sigma^2(y)}+\epsilon K(\theta_1,\theta_2)dy\bigg)\big|\theta_2-\theta_1\big| =: \overline{C}_2(x,\theta_1,\theta_2)\big|\theta_2-\theta_1\big|. \end{aligned}\tag{4.20}$$

□

Corollary 4.1. *The map $(x,\theta)\mapsto V_x^\epsilon(x,\theta)$ is continuous on $\mathbb{R}_+^2$.*

Introduce the inaction region $\mathcal{C}$ and the action region $\mathcal{S}$ as it follows:

$$\mathcal{C} := \{(x,\theta)\in\mathbb{R}_+^2 : V_x^\epsilon(x,\theta) > -c(x)\}\tag{4.21}$$

$$\mathcal{S} := \{(x,\theta)\in\mathbb{R}_+^2 : V_x^\epsilon(x,\theta) = -c(x)\},\tag{4.22}$$

and remember that

$$\beta_\epsilon(\theta) = \inf\{x\in\mathbb{R}_+ : V_x^\epsilon(x,\theta) = -c(x)\}.\tag{4.23}$$

We then have the following continuity result.

Theorem 4.1. *The map $\theta\mapsto\beta_\epsilon(\theta)$ is continuous.*

Proof. We split the proof into two steps.

Step 1: In this step we show that the map $\theta\mapsto\beta_\epsilon(\theta)$ is right-continuous. Fix $\theta\in\mathbb{R}_+$, and a sequence $\{\theta^n\}_{n\in\mathbb{N}}$ such that $\theta^n\searrow\theta$ as $n\uparrow\infty$. Since the map $\theta\mapsto\beta_\epsilon(\theta)$ is nonincreasing (cf. Lemma 4.1) we obtain that $\beta_\epsilon(\theta^n)\leq\beta_\epsilon(\theta)$, for any $n\in\mathbb{N}$, which implies $\beta_\epsilon(\theta^+) := \lim_{n\uparrow\infty}\beta_\epsilon(\theta^n)\leq\beta_\epsilon(\theta)$. It remains to show that $\beta_\epsilon(\theta)\leq\beta_\epsilon(\theta^+)$. To that end, we observe that $(\theta^n,\beta_\epsilon(\theta^n))\in\mathcal{S}$, for any $n\in\mathbb{N}$, and $(\theta^n,\beta_\epsilon(\theta^n))\to(\theta,\beta_\epsilon(\theta^+))$ as $n\uparrow\infty$. Thanks to continuity of $(x,\theta)\mapsto V_x^\epsilon(x,\theta)$ (cf. Corollary 4.1) we know that $\mathcal{S}$ is closed, hence $(\theta,\beta_\epsilon(\theta^+))\in\mathcal{S}$. Now, (4.23) gives $\beta_\epsilon(\theta)\leq\beta_\epsilon(\theta^+)$.

Step 2: Now we prove that map $\theta\mapsto\beta_\epsilon(\theta)$ is left-continuous. In order to do this, we borrow ideas from [29] (see also [36]). Arguing by contradiction, we assume that there exists $\theta_0\in\Theta$ such that $\beta_\epsilon(\theta_0)<\beta_\epsilon(\theta_0^-)$, where $\beta_\epsilon(\theta_0^-) := \lim_{\delta\downarrow 0}\beta_\epsilon(\theta_0-\delta)$. The limit exists due to the monotonicity of $\theta\mapsto\beta_\epsilon(\theta)$ (cf. Lemma 4.1). Then, we can choose $x_1,x_2\in\mathbb{R}_+$ such that $\beta_\epsilon(\theta_0)<x_1<x_2<\beta_\epsilon(\theta_0^-)$ and $\theta_1<\theta_0$. We define a rectangular domain denoted by $\mathcal{R}$ with vertices $(x_1,\theta_1),(x_1,\theta_0),(x_2,\theta_1)$ and (x_2,θ_0). Notice that $\mathcal{R}\subset\mathcal{C}$ and $[x_1,x_2]\times\{\theta_0\}\subset\mathcal{S}$. From (3.5) we know that V^ϵ satisfies

$$\begin{cases}\mathcal{L}^\epsilon V^\epsilon(x,\theta)+\pi(x,\theta)=\lambda^\epsilon(\theta), & (x,\theta)\in[x_1,x_2]\times[\theta_1,\theta_0),\\ V_x^\epsilon(x,\theta_0)=-c(x), & x\in[x_1,x_2].\end{cases}\tag{4.24}$$

Denote by $C_c^\infty((x_1,x_2))$ the set of functions with infinitely many continuous derivatives and compact support in (x_1,x_2). Pick arbitrary $\psi\in C_c^\infty((x_1,x_2))$ such that $\psi\geq 0$ and $\int_{x_1}^{x_2}\psi(x)dx>0$, and, for $\theta\in[\theta_1,\theta_0)$, multiply the first equation in (4.24) by ψ and integrate both sides over (x_1,x_2). This gives

$$\int_{x_1}^{x_2}\big(\mathcal{L}^\epsilon V^\epsilon(x,\theta)+\pi(x,\theta)\big)\psi(x)dx = \int_{x_1}^{x_2}\lambda^\epsilon(\theta)\psi(x)dx.\tag{4.25}$$

Rearranging terms and using integration by parts on the left-hand side we obtain

$$-\int_{x_1}^{x_2}\Big(\frac{1}{2}\sigma^2(x)\psi(x)\Big)_x V_x^\epsilon(x,\theta)dx = \int_{x_1}^{x_2}\big(\lambda^\epsilon(\theta) - b(x)V_x^\epsilon(x,\theta) \\ -\pi(x,\theta) + \frac{\epsilon}{2}\sigma^2(x)(V_x^\epsilon)^2(x,\theta)\big)\psi(x)dx. \tag{4.26}$$

From Proposition 4.1 we know that the map $\theta \mapsto V_x^\epsilon(x,\theta),\ x\in\mathbb{R}_+$, is continuous. Hence, taking limits as $\theta\uparrow\theta_0$, by invoking the dominated convergence theorem, we obtain

$$-\int_{x_1}^{x_2}\Big(\frac{1}{2}\sigma^2(x)\psi(x)\Big)_x V_x^\epsilon(x,\theta_0)dx = \int_{x_1}^{x_2}\big(\lambda^\epsilon(\theta_0) - b(x)V_x^\epsilon(x,\theta_0) \\ -\pi(x,\theta_0) + \frac{\epsilon}{2}\sigma^2(x)(V_x^\epsilon)^2(x,\theta_0)\big)\psi(x)dx. \tag{4.27}$$

Since now $V_x^\epsilon(x,\theta_0) = -c(x),\ x\in[x_1,x_2]$, recalling (3.6) and that $\lambda^\epsilon(\theta) = \ell^\epsilon(\beta_\epsilon(\theta),\theta)$, applying again integration by parts on the left-hand side and rearranging the terms, we obtain

$$\int_{x_1}^{x_2}\big(\ell^\epsilon(\beta_\epsilon(\theta_0),\theta_0) - \ell^\epsilon(x,\theta_0)\big)\psi(x)dx = 0. \tag{4.28}$$

However, the left-hand side of (4.28) is strictly negative by Assumption 3.1-(3.7). Hence, we have a contradiction. □

4.2. **Existence and Uniqueness of the Ergodic MFG Equilibrium.**

Proposition 4.2. *For $\epsilon>0$, the following hold:*

(1) *For any $\theta\in\mathbb{R}_+$, there exists a stationary distribution of $(X_t^{\xi^*(\theta)})_{t\geq 0}$ under $\mathbb{Q}^*(\theta)$, denoted by $\nu^{\theta,\epsilon}\in\mathcal{P}(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$, and its density, denoted by $m^{\theta,\epsilon}$, is such that*

$$m^{\theta,\epsilon}(x) = \frac{2}{\nu^{\theta,\epsilon}((0,\beta_\epsilon(\theta)])\sigma^2(x)}\exp\Big(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(x)}{\sigma^2(y)}dy + 2\epsilon\int_x^{\beta_\epsilon(\theta)} V_x^\epsilon(y,\theta)dy\Big)\mathbf{1}_{(0,\beta_\epsilon(\theta)]}(x), \tag{4.29}$$

where

$$\nu^{\theta,\epsilon}((0,\beta_\epsilon(\theta)]) := \int_0^{\beta_\epsilon(\theta)}\frac{2}{\sigma^2(x)}\exp\Big(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(x)}{\sigma^2(y)}dy + 2\epsilon\int_x^{\beta_\epsilon(\theta)} V_x^\epsilon(y,\theta)dy\Big)dx$$

(2) *The map $\theta\mapsto\nu^{\theta,\epsilon}$ is continuous.*

Proof. Let $x\in\mathbb{R}_+$ and $\theta\in\mathbb{R}_+$. From Proposition 3.4, we know that under $\mathbb{Q}^*(\theta)\in\widehat{\mathcal{Q}}(x)$, with $\frac{d\mathbb{Q}^*(\theta)}{d\mathbb{P}}\big|_{\mathcal{F}_t} = -\epsilon\sigma(X_t^{\xi^*(\theta)})V_x^\epsilon(X_t^{\xi^*(\theta)},\theta)$ $\mathbb{P}$-a.s. (cf. Theorem 3.2), the process $(X_t^{\xi^*(\theta)})_{t\geq 0}$ evolves as

$$dX_t^{\xi^*(\theta)} = \big(b(X_t^{\xi^*(\theta)}) - \epsilon\sigma^2(X_t^{\xi^*(\theta)})V_x^\epsilon(X_t^{\xi^*(\theta)},\theta)\big)dt + \sigma(X_t^{\xi^*(\theta)})dW_t^{\mathbb{Q}^*(\theta)} - d\xi_t^*(\theta), \tag{4.30}$$

with $X_0^{\xi^*(\theta)} = x$, and $(X^{\xi^*(\theta)},\xi^*(\theta))$ solving $\mathbf{SP}(x,\beta_\epsilon(\theta);\mathbb{Q}^*(\theta),-\epsilon\sigma(\cdot)V_x^\epsilon(\cdot,\theta))$ (cf. Definition 3.1). From Proposition 3.3, there exists $M := M(\theta) > 0$ such that $|V_x^\epsilon(x,\theta)|\leq M(\theta)$ for any $x\in[0,\infty)$ (see (3.17)), so that

$$\begin{aligned}\nu^{\theta,\epsilon}((0,\beta_\epsilon(\theta)]) &= \int_0^{\beta_\epsilon(\theta)}\frac{2}{\sigma^2(x)}\exp\Big(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(x)}{\sigma^2(y)}dy + 2\epsilon\int_x^{\beta_\epsilon(\theta)} V_x^\epsilon(y,\theta)dy\Big)dx \\ &\leq \int_0^{\beta_\epsilon(\theta)}\frac{2}{\sigma^2(x)}\exp\Big(-\int_x^{\beta_\epsilon(\theta)}\frac{2b(x)}{\sigma^2(y)}dy + 2\epsilon M\beta_\epsilon(\theta)\Big)dx \\ &=: \exp(2\epsilon M\beta_\epsilon(\theta))m^{\mathbb{P}}((0,\beta_\epsilon(\theta)]) < \infty,\end{aligned} \tag{4.31}$$

by Assumption 2.2.

Then, from Section 36 of Chapter II in [10] the process $(X_t^{\xi^*(\theta)})_{t\geq 0}$ is ergodic and has invariant measure $\nu^{\theta,\epsilon} \in \mathcal{P}(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ with

$$\nu^{\theta,\epsilon}((0,x)) = \int_0^x m^{\theta,\epsilon}(y)dy, \quad x \in (0, \beta_\epsilon(\theta)]. \tag{4.32}$$

Claim 2 follows from Proposition 4.1 and Theorem 4.1. □

We are now in the position to prove the main result of this section. To this end, recall Assumption 2.4 we introduce the operator $\mathcal{T} : \mathbb{R}_+ \to \mathbb{R}_+$, as

$$\mathcal{T}\theta := F\left(\int_{\mathbb{R}_+} f(x)\nu^{\theta,\epsilon}(dx)\right) = F(\langle f, \nu^{\theta,\epsilon}\rangle), \tag{4.33}$$

where $\langle f, \nu^{\theta,\epsilon}\rangle := \int_{\mathbb{R}_+} f(x)\nu^{\theta,\epsilon}(dx)$. Thanks to the previous results, we can now prove the existence and uniqueness of a stationary mean-field equilibrium as in Definition 2.3.

Theorem 4.2. *For any $\epsilon > 0$, there exists a unique $\theta^\epsilon \in \mathbb{R}_+$ such that $\theta^\epsilon = \mathcal{T}\theta^\epsilon$.*

Proof. Let $\epsilon > 0$. We divide the proof into three steps.

Step 1: Set of relevant θ. Let $x \in \mathbb{R}_+$ and $\theta \in \mathbb{R}_+$. We start by obtaining a lower bound for $\mathcal{T}\theta$, which is uniform with respect to θ. For $\underline{\beta}_\epsilon \in \mathbb{R}_+$ as in Lemma 4.2, and arguing as in Proposition 3.1, we can find $\underline{V}^\epsilon \in C^2(\mathbb{R}_+)$ to be the unique classical solution to the problem

$$\begin{cases} \frac{1}{2}\sigma^2(x)\underline{V}^\epsilon_{xx}(x) + b(x)\underline{V}^\epsilon_x(x) - \frac{\epsilon}{2}\sigma^2(x)\big(\underline{V}^\epsilon_x\big)^2(x) = \underline{\lambda}^\epsilon - \kappa(x), \quad x < \underline{\beta}_\epsilon, \\ \underline{V}^\epsilon_x(x) = -c(x), \quad x \geq \underline{\beta}_\epsilon. \end{cases} \tag{4.34}$$

Recall $\mathbb{Q}^*(\theta)$ (cf. Theorem 3.2). Then, Proposition 3.4 allows to construct an $\mathbb{F}$-adapted pair $(Y^\zeta, \zeta) \in \mathcal{D}[0,\infty) \times \mathcal{A}$ such that

$$dY_t^\zeta = (b(Y_t^\zeta) - \epsilon\sigma^2(Y_t^\zeta)\underline{V}^\epsilon_x(Y_t^\zeta))dt + \sigma(Y_t^\zeta)dW_t^{\mathbb{Q}^*(\theta)} - d\zeta_t, \quad Y_0^\zeta = x, \tag{4.35}$$

and (Y^ζ, ζ) solves $\mathbf{SP}(x, \underline{\beta}_\epsilon; \mathbb{Q}^*(\theta), -\epsilon\sigma\underline{V}^\epsilon_x)$. Notice that, while Y^ζ is independent of θ, its expectation under $\mathbb{Q}^*(\theta)$ is not. However, this is easily fixed. We define a new complete probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{Q}})$ supporting a Brownian motion $(\widehat{W}_t)_{t\geq 0}$, let $(\widehat{\mathcal{F}}_t^o)_{t\geq 0}$ be the filtration generated by Brownian motion $\widehat{W}$, and denote by $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{t\geq 0}$ its augmentation with the $\widehat{\mathbb{Q}}$-null sets. Hence, we introduce

$$\widehat{\mathcal{A}} := \{(\widehat{\xi}_t)_{t\geq 0},\ \widehat{\mathbb{F}}\text{-adapted, nondecreasing, left-continuous and such that } \widehat{\xi}_0 = 0,\ \widehat{\mathbb{Q}}\text{-a.s.}\}. \tag{4.36}$$

Thanks to Lemma 5.5 in [30], since $\zeta \in \mathcal{A}$ we can find $\widehat{\zeta} \in \widehat{\mathcal{A}}$ that is $(\widehat{\mathcal{F}}_t^o)_{t\geq 0}$-predictable and such that $\mathrm{Law}_{\mathbb{Q}^*(\theta)}(W^{\mathbb{Q}^*(\theta)}, \zeta) = \mathrm{Law}_{\widehat{\mathbb{Q}}}(\widehat{W}, \widehat{\zeta})$. Therefore, from Lemma 5.6 in [30] we have that

$$\mathrm{Law}_{\mathbb{Q}^*(\theta)}(W^{\mathbb{Q}^*(\theta)}, Y^\zeta, \zeta) = \mathrm{Law}_{\widehat{\mathbb{Q}}}(\widehat{W}, \widehat{Y}^{\widehat{\zeta}}, \widehat{\zeta}), \tag{4.37}$$

where $(\widehat{Y}^{\widehat{\zeta}})$ is the unique strong solution on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{Q}}, \widehat{\mathbb{F}})$ to

$$d\widehat{Y}_t^{\widehat{\zeta}} = \big(b(\widehat{Y}_t^{\widehat{\zeta}}) - \epsilon\sigma^2(\widehat{Y}_t^{\widehat{\zeta}})\underline{V}^\epsilon_x(\widehat{Y}_t^{\widehat{\zeta}})\big)dt + \sigma(\widehat{Y}_t^{\widehat{\zeta}})d\widehat{W}_t - d\widehat{\zeta}_t, \quad \widehat{Y}_0^{\widehat{\zeta}} = x, \tag{4.38}$$

subject to $(\widehat{Y}^{\widehat{\zeta}}, \widehat{\zeta})$ uniquely solving $\mathbf{SP}(x, \underline{\beta}_\epsilon; \widehat{\mathbb{Q}}, -\epsilon\sigma\underline{V}^\epsilon_x)$. Hence, we have that

$$\mathbb{E}_x^{\widehat{\mathbb{Q}}}\big[f(\widehat{Y}_t^{\widehat{\zeta}})\big] = \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[f(Y_t^\zeta)\big], \text{ for any } t \geq 0. \tag{4.39}$$

Furthermore, arguing as in Proposition 4.2, we can show that $(\widehat{Y}_t^{\widehat{\zeta}})_{t\geq 0}$ admits a unique stationary distribution denoted by $\underline{\nu}^\epsilon \in \mathcal{P}(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Thus, by ergodicity of $\widehat{Y}^{\widehat{\zeta}}$ (see Section 36 of Chapter II in [10]), we obtain that

$$\langle f, \underline{\nu}^\epsilon\rangle = \lim_{T\uparrow\infty} \frac{1}{T}\int_0^T \mathbb{E}_x^{\widehat{\mathbb{Q}}}\big[f(\widehat{Y}_t^{\widehat{\zeta}})\big]dt, \tag{4.40}$$

and we define

$$\underline{\theta}_1^\epsilon := F\big(\langle f, \underline{\nu}^\epsilon\rangle\big) \quad \text{and} \quad \overline{\theta}_1^\epsilon := F\big(f(\beta_\epsilon(\underline{\theta}_1^\epsilon))\big). \tag{4.41}$$

It is then clear that

$$\underline{\theta}_1^\epsilon := F\big(\langle f, \underline{\nu}^\epsilon\rangle\big) \le F\big(f(\underline{\beta}_\epsilon)\big) \le F\big(f(\beta_\epsilon(\underline{\theta}_1^\epsilon)\big) = \overline{\theta}_1^\epsilon, \tag{4.42}$$

where the first inequality follows from Assumption 2.4-(1) and the second inequality is due to Lemma 4.2.

For $\theta \in [\underline{\theta}_1^\epsilon, \overline{\theta}_1^\epsilon]$, we know that $\beta_\epsilon(\overline{\theta}_1^\epsilon) \le \beta_\epsilon(\theta) \le \beta_\epsilon(\underline{\theta}_1^\epsilon)$ (cf. Lemma 4.1). Furthermore, it holds

$$V_x^\epsilon(x,\theta) = \phi_{\beta_\epsilon(\theta)}(x,\theta) \le \phi_{\beta_\epsilon(\underline{\theta}_1^\epsilon)}(x,\theta) \le \phi_{\beta_\epsilon(\underline{\theta}_1^\epsilon)}(x,\overline{\theta}_1^\epsilon) =: \widehat{V}_x^\epsilon(x), \quad x \in (0, \beta_\epsilon(\theta)], \tag{4.43}$$

where the first inequality follows from Proposition A.2 and the second follows from Proposition A.1. Additionally, $\widehat{V}_x^\epsilon$ is the unique solution to (3.10) for $\beta = \beta_\epsilon(\underline{\theta}_1^\epsilon)$, $\theta = \overline{\theta}_1^\epsilon$ and $\gamma = 0$. We introduce the $\mathbb{F}$-adapted process $(Z_t)_{t\ge 0}$ with dynamics

$$dZ_t = (b(Z_t) - \epsilon\sigma^2(Z_t)\widehat{V}_x^\epsilon(Z_t))dt + \sigma(Z_t)dW_t^{\mathbb{Q}^*(\theta)}, \quad Z_0 = x. \tag{4.44}$$

Thanks to the regularity of $\widehat{V}^\epsilon$ (cf. Proposition 3.1), equation (4.44) admits a unique strong solution. Arguing as in Proposition 3.4, we can find a pair $(Z^{\xi(\underline{\theta}_1^\epsilon)}, \xi(\underline{\theta}_1^\epsilon)) \in \mathcal{D}[0,\infty) \times \mathcal{A}$ that uniquely solves $\mathbf{SP}(x, \beta_\epsilon(\overline{\theta}_1^\epsilon); \mathbb{Q}^*(\theta), -\epsilon\sigma\widehat{V}_x^\epsilon)$. Consequently, given that f is increasing, Proposition A.2 gives us

$$\mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[f(Z^{\xi(\underline{\theta}_1^\epsilon)})\big] \le \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[f(X_t^{\xi^*(\theta)})\big], \tag{4.45}$$

where $(X_t^{\xi^*(\theta)})_{t\ge 0}$ is the strong solution to

$$dX_t^{\xi^*(\theta)} = (b(X_t^{\xi^*(\theta)}) - \epsilon\sigma^2(X_t^{\xi^*(\theta)})V_x^\epsilon(X_t^{\xi^*(\theta)},\theta))dt + \sigma(X_t^{\xi^*(\theta)})dW_t^{\mathbb{Q}^*(\theta)} - d\xi_t^*(\theta), \tag{4.46}$$

with $\xi^*(\theta)$ as in (3.19) (cf. Proposition 3.4). Furthermore, using again Lemma 5.5 and Lemma 5.6 from [30] we can find $(\widehat{Z}^{\widehat{\xi}(\underline{\theta}_1^\epsilon)}, \widehat{\xi}(\underline{\theta}_1^\epsilon)) \in \mathcal{D}[0,\infty)\times\widehat{\mathcal{A}}$ such that $(\widehat{Z}^{\widehat{\xi}(\underline{\theta}_1^\epsilon)}, \widehat{\xi}(\underline{\theta}_1^\epsilon))$ solves $\mathbf{SP}(x, \beta_\epsilon(\overline{\theta}_1^\epsilon); \widehat{\mathbb{Q}}, -\epsilon\sigma\widehat{V}_x^\epsilon)$, where $\widehat{Z}^{\widehat{\xi}(\underline{\theta}_1^\epsilon)}$ is the unique strong solution to

$$d\widehat{Z}_t^{\widehat{\xi}(\underline{\theta}_1^\epsilon)} = (b(\widehat{Z}_t^{\widehat{\xi}(\underline{\theta}_1^\epsilon)}) - \epsilon\sigma^2(\widehat{Z}_t^{\widehat{\xi}(\underline{\theta}_1^\epsilon)})\widehat{V}_x^\epsilon(\widehat{Z}_t^{\widehat{\xi}(\underline{\theta}_1^\epsilon)}))dt + \sigma(\widehat{Z}_t^{\widehat{\xi}(\underline{\theta}_1^\epsilon)})dW_t^{\widehat{\mathbb{Q}}} - d\widehat{\xi}_t(\underline{\theta}_1^\epsilon) \tag{4.47}$$

and

$$\mathbb{E}_x^{\widehat{\mathbb{Q}}}\big[f(\widehat{Z}_t^{\widehat{\xi}(\underline{\theta}_1^\epsilon)})\big] = \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[f(Z_t^{\xi(\underline{\theta}_1^\epsilon)})\big], \text{ for any } t \ge 0. \tag{4.48}$$

We claim that $(\widehat{Z}_t^{\widehat{\xi}(\underline{\theta}_1^\epsilon)})_{t\ge 0}$ admits a stationary distribution under $\widehat{\mathbb{Q}}$. Indeed, arguing as in the proof of Proposition 4.2, we can show that there exists unique $\widehat{\nu}^\epsilon \in \mathcal{P}(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Hence, from ergodicity of $\widehat{Z}^{\widehat{\xi}(\underline{\theta}_1^\epsilon)}$, (4.45) and (4.48) we have

$$\langle f, \widehat{\nu}^\epsilon\rangle = \lim_{T\uparrow\infty}\frac{1}{T}\int_0^T \mathbb{E}_x^{\widehat{\mathbb{Q}}}\big[f(\widehat{Z}_t^{\widehat{\xi}(\underline{\theta}_1^\epsilon)})\big]dt \le \lim_{T\uparrow\infty}\frac{1}{T}\int_0^T \mathbb{E}_x^{\mathbb{Q}^*(\theta)}\big[f(X_t^{\xi^*(\theta)})\big]dt = \langle f, \nu^{\theta,\epsilon}\rangle, \tag{4.49}$$

which in turn, by monotonicity of F, leads to

$$\underline{\theta}^\epsilon := F(\langle f, \widehat{\nu}^\epsilon\rangle) \le \mathcal{T}\theta. \tag{4.50}$$

To find an upper bound for $\mathcal{T}\theta$, is easier and we proceed as follows. Since $X_t^{\xi^*(\theta)} \in (0, \beta_\epsilon(\theta)]$, $\mathbb{Q}_x^*(\theta)$-a.s. and thanks to the monotonicity of f and F (see Assumption 2.4-(1))), we obtain

$$\mathcal{T}\theta = F(\langle f, \nu^{\theta,\epsilon}\rangle) \le F(f(\beta_\epsilon(\theta)) \le F(f(\beta_\epsilon(\underline{\theta}^\epsilon))) =: \overline{\theta}^\epsilon, \tag{4.51}$$

for any $\theta \ge \underline{\theta}^\epsilon$. In the last inequality we have used the fact that the map $\theta \mapsto \beta_\epsilon(\theta)$ is nonincreasing (cf. Lemma 4.1). Thus, combining (4.50) and (4.51) we conclude that any potential fixed point of $\mathcal{T}$ must lie in the convex, compact set

$$K^\epsilon := [\underline{\theta}^\epsilon, \overline{\theta}^\epsilon] \subset \mathbb{R}_+. \tag{4.52}$$

Step 2: Continuity of $\mathcal{T}$. We define the map $\mathcal{T}_1 : K^\epsilon \to \mathcal{P}(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ by

$$\mathcal{T}_1\theta = \nu^{\theta,\epsilon}, \quad \theta \in K^\epsilon. \tag{4.53}$$

The map is well-defined and continuous thanks to Proposition 4.2-(2). Next, we denote by $\mathcal{T}_2$: $\mathcal{P}(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \to K^\epsilon$ the map

$$\mathcal{T}_2\nu := F\bigg(\int_{\mathbb{R}_+} f(x)\nu(dx)\bigg). \tag{4.54}$$

Since the functions f and F are continuous and the probability measures have compact support, the map $\mathcal{T}_2$ is clearly continuous. Concluding, the map $\mathcal{T} := \mathcal{T}_2 \circ \mathcal{T}_1 : K^\epsilon \to K^\epsilon$ is continuous in the convex compact set K^ϵ and, by Schauder-Tychonof fixed-point theorem (Corollary 17.56 in [4]) there exists $\theta^\epsilon \in K^\epsilon$ such that $\mathcal{T}\theta^\epsilon = \theta^\epsilon$.

Step 3: Uniqueness. Let $\theta^\epsilon \in K^\epsilon$ be the fixed-point of $\mathcal{T}$ and let $\tilde{\theta}^\epsilon \in K^\epsilon$ be another fixed-point of $\mathcal{T}$ such that $\theta^\epsilon \neq \tilde{\theta}^\epsilon$. Without loss of generality, we assume that $\theta^\epsilon > \tilde{\theta}^\epsilon$. Then, by monotonicity of $\theta \mapsto \beta_\epsilon(\theta)$ (cf. Lemma 4.1), we have that $\beta_\epsilon(\theta^\epsilon) \leq \beta_\epsilon(\tilde{\theta}^\epsilon)$, and mimicking Step 1 we obtain that

$$\theta^\epsilon = \mathcal{T}\theta^\epsilon \leq \mathcal{T}\tilde{\theta}^\epsilon = \tilde{\theta}^\epsilon, \tag{4.55}$$

which leads to a contradiction. □

5. A Case Study: Optimal Extraction of a Natural Resource

This section is numerically illustrates our previous findings in a mean-field game of optimal extraction of natural resources under Knightian uncertainty. Inspired from [24] (see also [59]), we assume that the (controlled) natural resource evolves (under $\mathbb{P}$) according to the following dynamics

$$dX_t^\xi = \alpha(\kappa - X_t^\xi)dt + \sigma X_t^\xi dW_t^{\mathbb{P}} - d\xi_t, \quad X_t^\xi = x, \tag{5.1}$$

where $\kappa > 0$ is the stationary average resource based on the capacity of the environment, $\alpha > 0$ denotes the rate of extinction, $\sigma > 0$ the rate of fluctuations and $\xi \in \mathcal{A}_e(x)$ represents the cumulative extraction strategy. At this point, it is important to show that the (uncontrolled) natural resource cannot extinct, in other words, we have to show that 0 is unattainable. To that end, for arbitrary $x_0 > 0$ we have

$$S_x^{\mathbb{P}}(x) = \exp\bigg(-\int_{x_0}^{x} \frac{2\alpha(\kappa - y)}{\sigma^2 y^2}dy\bigg) \sim \exp\bigg(\frac{2\alpha\kappa}{\sigma^2 x} + \frac{2\alpha}{\sigma^2}\ln x\bigg), \tag{5.2}$$

which readily implies that the Assumption 2.2 is satisfied.

Following Remark 2.3, the representative firm faces constant cost of extraction $c > 0$, uses an isoelastic demand function in the style of Dixit-Stiglitz-Spence preferences (see also [35]), which results into an instantaneous profit $\pi(x,\theta) = x^\delta(\theta^{-(1+\delta)} + \eta)$, depending on a fixed price $\eta > 0$ and on a price index $\theta^{-(1+\delta)}$, which, at equilibrium, is such that

$$\theta = \bigg(\int_{\mathbb{R}_+} x^\delta \nu_\infty(dx)\bigg)^{1/\delta}, \tag{5.3}$$

where ν_∞ is the stationary distribution of the optimally controlled resource stock.

From Remarks 2.1 and 2.2 we obtain that Assumptions 2.1, 2.2, 2.3, 2.4, 3.1 and 4.1 are satisfied. Hence, there exists a unique equilibrium of the ergodic MFG $(\beta_\epsilon(\theta^\epsilon), \theta^\epsilon, \mathbb{Q}^*(\theta^\epsilon))$ which satisfies Definition 2.3. Moreover, there exists $\phi_{\beta_\epsilon(\theta^\epsilon)}(\cdot, \theta^\epsilon) \in C^1(\mathbb{R}_+)$ such that

$$\begin{aligned} \frac{1}{2}\sigma^2 x^2 (\phi_{\beta_\epsilon(\theta^\epsilon)})_x(x,\theta^\epsilon) + (\kappa - \alpha x)\phi_{\beta_\epsilon(\theta^\epsilon)}(x,\theta^\epsilon) - \frac{\epsilon}{2}\sigma^2 x^2 (\phi_{\beta_\epsilon(\theta^\epsilon)})^2(x,\theta^\epsilon) \\ = \lambda^\epsilon(\theta^\epsilon) - x^\delta((\theta^\epsilon)^{-(1+\delta)} + \eta), \quad x < \beta_\epsilon(\theta^\epsilon) \\ \phi_{\beta_\epsilon(\theta^\epsilon)}(x,\theta^\epsilon) = -c, \quad x \geq \beta_\epsilon(\theta^\epsilon), \end{aligned} \tag{5.4}$$

where $\lambda^\epsilon(\theta^\epsilon) = -c(1-\beta_\epsilon(\theta^\epsilon)) + (\beta_\epsilon(\theta^\epsilon))^\delta((\theta^\epsilon)^{-(1+\delta)}+\eta) - \frac{\epsilon}{2}\sigma^2(\beta_\epsilon(\theta^\epsilon))^2$ (see (3.20) in Theorem 3.2).

Given that the equilibrium cannot be determined explicitly, we introduce a policy iteration algorithm for its evaluation. The policy iteration method introduced by Bellman [8] is an algorithm to solve numerically Hamilton-Jacobi-Bellman equations. Recently, it has been generalized to the case of mean-field games and we refer to Cacace et al. [11] and Camilli and Tang [14]. Our algorithm is inspired from [34] and it is described in the following table.

Algorithm 1: Policy Iteration Algorithm

Input: $\theta^{(0)} = \underline{\theta}^\epsilon$ and $\epsilon << 1$.

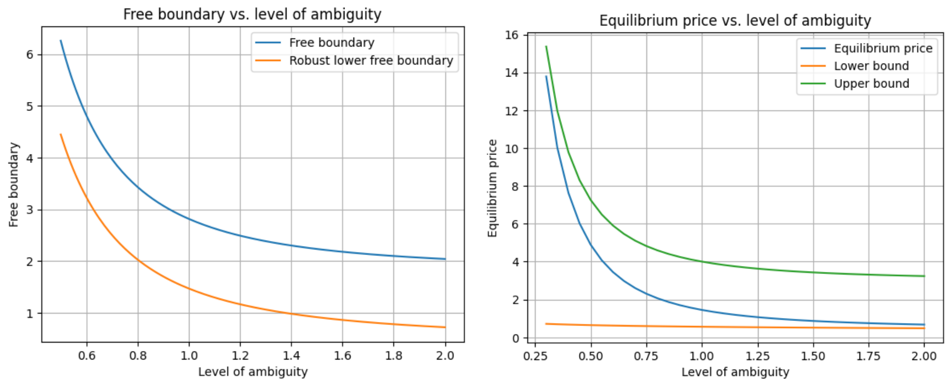
for $n = 0$ **to** $N-1$ **do**

- **Input:** $\beta^{(0,n)} = \underline{\widehat{x}}_\epsilon(\theta^{(n)})$. Find $\phi^{(0,n)}(x,\theta^{(n)}) \in C^1(\mathbb{R}_+)$ solution to (3.10) for $\beta = \beta^{(0,n)}$ and $\gamma = 0$.
- **for** $k = 0$ **to** $N-1$ **do**
  - Calculate
$$\beta^* = \max\{x \in (\widehat{x}_\epsilon(\theta^{(n)}), \beta^{(k,n)}) : \phi^{(k)}(x,\theta^{(n)}) = -c+\epsilon\}. \tag{5.5}$$
  - Find a $\phi^*(\cdot,\theta^{(n)}) \in C^1(\mathbb{R}_+)$ solution to the following equations:
$$\begin{aligned} \frac{1}{2}\sigma^2(x)(\phi^*)_x(x,\theta^{(n)}) + b(x)\phi^*(x,\theta^{(n)}) - \frac{\epsilon}{2}\sigma^2(x)(\phi^*)^2(x,\theta^{(n)}) \\ = \ell^\epsilon(\beta^*,\theta^{(n)}) - \pi(x,\theta^{(n)}), \quad x \le \beta^*, \end{aligned} \tag{5.6}$$
$$\phi^*(x,\theta^{(n)}) = -c, \quad x \ge \beta^*. \tag{5.7}$$
  - Update policy as follows
$$\beta^{(k+1,n)} = \begin{cases} \beta^*, & \text{if } \phi^*(x,\theta^{(n)}) \ge -c, \text{ for any } x \in (0,\beta^*], \\ \beta^{(k,n)}, & \text{otherwise.} \end{cases} \tag{5.8}$$
- Update mean-field equilibrium as follows
$$\theta^{(n+1)} = \begin{cases} F\big(\int_{\mathbb{R}_+} f(x)\nu^{\beta^{(K,n)}}(dx;\theta^{(n)})\big) =: \theta^*, & \text{if } \theta^* \in [\underline{\theta}^\epsilon, \overline{\theta}^\epsilon], \\ \theta^{(n)}, & \text{otherwise.} \end{cases} \tag{5.9}$$

return $(\beta^{(N,N)}, \theta^{(N)})$.

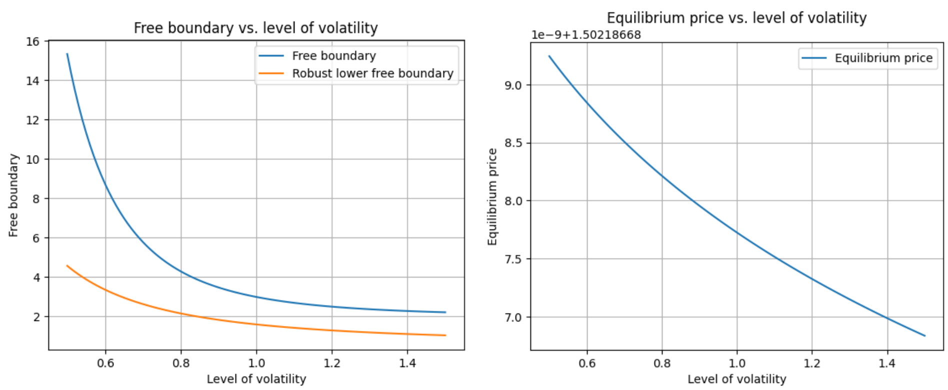
5.1. **Sensitivity Analysis.** In this section, using the policy iteration algorithm above, we numerically explore the sensitivity of the mean-field equilibrium with respect to the level of ambiguity ϵ and the level of volatility σ. Thereafter, we study the behavior of the stationary distribution of the optimally controlled harvested resource for different levels of ambiguity. In our numerical example, we set $\kappa = \alpha = \sigma_{\text{default}} = 1$, $\eta = c = 1$, $\delta = 0.6$ and $\epsilon_{\text{default}} = 1$.

5.1.1. *Level of Ambiguity* ϵ. The level of ambiguity quantifies the sensitivity of the representative firm to deviations from the real-world model $\mathbb{P}$. In other words, it measures the impact of the worst-case scenario model $\mathbb{Q}^*$. In Figure (1a) we see that as the level of ambiguity increases, the reflection boundary decreases. This happens due to the fact that, as the firm becomes more uncertain about the future growth rate of the resource it decides to extract more frequently instead of waiting until the natural resource will attain a higher level. This result is consistent with other studies on singular stochastic control problems under uncertainty (see for instance Theorem 5.1 in [27]). Moreover, we observe in Figure (1b) that also the equilibrium price decreases as the level of ambiguity increases.



(A) $\beta_\epsilon(\theta^\epsilon)$ vs. ϵ. (B) θ^ϵ vs. ϵ.

FIGURE 1. Comparative statics of equilibrium w.r.t. level of ambiguity ϵ.



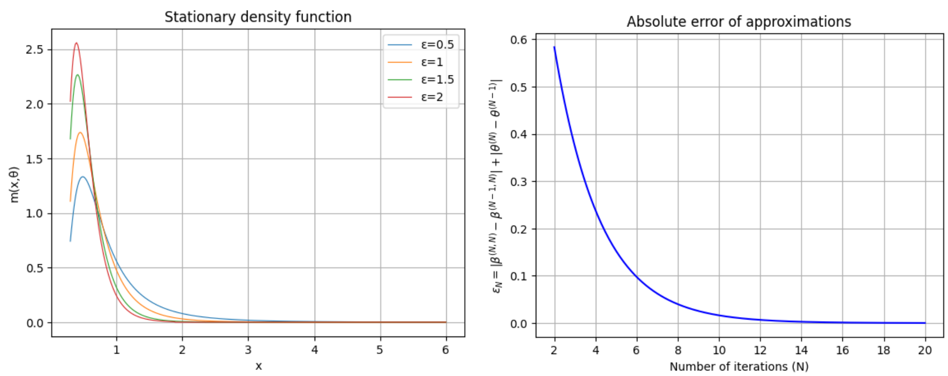
(A) $\beta_\epsilon(\theta^\epsilon)$ vs. σ. (B) θ^ϵ vs. σ.

FIGURE 2. Comparative statics of equilibria w.r.t. level of volatility σ.

Given constant demand, the increased frequency of extraction leads to higher supply, which in turn results in a lower price.

5.1.2. *Level of Volatility* σ. In our model, each firm extracts from a natural resource that is subject to exogenous risks, which are modeled through the Brownian motion $W^{\mathbb{P}}$. As illustrated in Figure (2a), an increase in volatility leads to a decrease in the level of extraction. Moreover, as depicted in Figure (2b), there is a reduction in the equilibrium price level as volatility increases. As volatility rises, natural resource exhibit higher sensitivity to fluctuations, causing firms to extract more in an effort to ensure long-term profitability. This increased extraction raises supply, which subsequently drives the price down.

5.1.3. *Effect of Ambiguity on firms' distribution size.* In continuation of Subsection 5.1.1, we examine the distribution of firms at varying levels of ambiguity. As shown in Figure (3a), higher levels of ambiguity lead to a concentration of firms at lower levels of natural resource extraction. This observation aligns with the findings of Subsection 5.1.1, allowing us to conclude that, at equilibrium, the majority of firms engage in low levels of extraction.



(A) Stationary density function for different levels of ambiguity.

(B) Absolute error of the algorithm for different number of iterations.

Appendix A. Some Technical Results

The proof of the next lemma is straightforward.

Lemma A.1. *Let $f \in C^1((\alpha,\beta))$. Fix $x \in (\alpha,\beta)$ and $\overline{y} := \sup\{y \in (\alpha,x) : f(y) = f(x)\}$ and $\underline{y} := \inf\{y \in (x,\beta) : f(y) = f(x)\}$, if they exist. If $f'(x) > 0$ (resp., < 0), then $f'(\overline{y}) \leq 0$ (resp., ≥ 0) and $f'(\underline{y}) \leq 0$ (resp., ≥ 0).*

Recall $\phi_{\cdot}(x,\theta)$ from Section 3.1. We then have the following Lemma from [27] (cf. Lemma 6).

Lemma A.2. *Recall $\widehat{x}_\epsilon(\theta)$ as in Assumption 3.1-(3.7) and let $\widehat{x}^\epsilon(\theta) \leq \alpha < \eta$. For any $x \in (0,\infty)$, one has $\phi_\alpha(x,\theta) \leq \phi_\eta(x,\theta)$.*

We are now in the position of proving the continuity of $\beta \mapsto \phi_\beta(x,\theta)$.

Lemma A.3 (Continuity of $\beta \mapsto \phi_\beta$)**.** *Recall ϕ_β from Section 3.1. For any fixed $\theta \in \mathbb{R}_+$ and fixed $\beta \geq \widehat{x}_\epsilon(\theta)$, we have*

$$\lim_{\delta \downarrow 0} |\phi_{\beta+\delta}(y,\theta) - \phi_\beta(y,\theta)| = 0, \quad \textit{for any } y < \beta. \tag{A.1}$$

Proof. Let $\theta \in \mathbb{R}_+$, $y \in (0,\beta)$ and $\delta \in (0,1)$. Given that $\phi_{\beta+\delta}$ and ϕ_β solve (3.10) with $\gamma = 0$ and $\beta+\delta$ and β, respectively, for $y \in [x,\beta]$ we have that

$$\begin{aligned}
&\big(\phi_{\beta+\delta}(y,\theta) - \phi_\beta(y,\theta)\big) \\
&\quad = \big(\phi_{\beta+\delta}(\beta,\theta) - \phi_\beta(\beta,\theta)\big) - \int_y^\beta \big(\phi_{\beta+\delta}(z,\theta) - \phi_\beta(z,\theta)\big)_x dy \\
&\quad = \big(\phi_{\beta+\delta}(\beta,\theta) - \phi_\beta(\beta,\theta)\big) - \big(\ell^\epsilon(\beta+\delta,\theta) - \ell^\epsilon(\beta,\theta)\big)\bigg(\int_y^\beta \frac{2}{\sigma^2(z)} dz\bigg) \\
&\qquad - \int_y^\beta \Big(\epsilon(\phi_{\beta+\delta}(z,\theta) + \phi_\beta(z,\theta)) - \frac{2b(z)}{\sigma^2(z)}\Big)\big(\phi_{\beta+\delta}(z,\theta) - \phi_\beta(z,\theta)\big) dz.
\end{aligned} \tag{A.2}$$

From Lemma A.2 we know that $\phi_\beta(z,\theta) \leq \phi_{\beta+\delta}(z,\theta) \leq \phi_{\beta+1}(z,\theta)$ for any $z \in [y,\beta]$, while from Proposition 3.3 we have that $\sup_{x\geq 0} \big|\phi_{\beta+1}(x,\theta)\big| \leq M(\theta)$, for some $M(\theta) > 0$. Hence, we have that

$$\begin{aligned}
&\big|\phi_{\beta+\delta}(y,\theta) - \phi_\beta(y,\theta)\big| \\
&\quad \leq \big|\phi_{\beta+\delta}(\beta,\theta) - \phi_\beta(\beta,\theta)\big| + \big|\ell^\epsilon(\beta+\delta) - \ell^\epsilon(\beta,\theta)\big|\bigg(\int_y^\beta \frac{2}{\sigma^2(z)} dy\bigg) \\
&\qquad + \int_y^\beta \bigg(2\epsilon M(\theta) + \frac{2|b(z)|}{\sigma^2(z)}\bigg)\big|\phi_{(\beta+\delta}(z,\theta) - \phi_\beta(z,\theta)\big| dz.
\end{aligned} \tag{A.3}$$

An application of Grönwall's inequality yields that

$$
\begin{aligned}
\text{(A.4)}\qquad & \big|\phi_{\beta+\delta}(y,\theta)-\phi_\beta(y,\theta)\big| \\
&\le \bigg(\big|\phi_{\beta+\delta}(\beta,\theta)-\phi_\beta(\beta,\theta)\big| + \bigg(\int_y^\beta \frac{2}{\sigma^2(z)}dz\bigg)\big|\ell^\epsilon(\beta+\delta,\theta)-\ell^\epsilon(\beta,\theta)\big|\bigg) \\
&\quad \cdot \exp\bigg(\int_y^\beta \bigg(2\epsilon M(\theta)+\frac{2|b(z)|}{\sigma^2(z)}\bigg)dz\bigg).
\end{aligned}
$$

It remains to show that the right-hand side of (A.4) vanishes as $\delta \downarrow 0$. Let $(\delta_n)_{n\in\mathbb{N}}$ be an arbitrarily fixed sequence such that $\delta_n \downarrow 0$, then introduce $\alpha_n := \phi_{\beta+\delta_n}(\beta,\theta)-\phi_\beta(\beta,\theta)$. From Lemma A.2 one has $\alpha_n \ge \alpha_{n+1}$ and $\alpha_n \ge 0$ for any $n\in\mathbb{N}$, so that $\lim_{n\uparrow\infty}\alpha_n = \inf_{n\in\mathbb{N}}\alpha_n = 0$. Recalling that $\beta \mapsto \ell^\epsilon(\beta,\theta),\ \theta\in\mathbb{R}_+$ is continuous (cf. Assumption 3.1) and taking limits in the right-hand side of (A.4) we obtain

$$
\text{(A.5)}\qquad \lim_{n\uparrow\infty}\bigg(\big|\phi_{\beta+\delta_n}(\beta,\theta)-\phi_\beta(\beta,\theta)\big| + \bigg(\int_y^\beta \frac{2}{\sigma^2(z)}dz\bigg)\big|\ell^\epsilon(\beta+\delta_n,\theta)-\ell^\epsilon(\beta,\theta)\big|\bigg)=0,
$$

which, by (A.4), allows to conclude. □

Proposition A.1 (Comparison principle wrt θ)**.** *Recall $\phi_\cdot(x,\theta)$ as in Section 3.1. For any $\beta\in\mathbb{R}_+$ and any $\theta_1,\theta_2\in\mathbb{R}_+$ with $\theta_1\le\theta_2$, the following hold:*

(1) $\phi_\beta(x,\theta_1)\le\phi_\beta(x,\theta_2)$ *for any* $x\in\mathbb{R}_+$.

(2) *Let* $\underline{\phi}_\beta\in C^1(\mathbb{R}_+)$ *be the unique solution to*

$$
\begin{aligned}
\text{(A.6)}\qquad \frac{1}{2}\sigma^2(x)\partial_x\underline{\phi}_\beta(x)+b(x)\underline{\phi}_\beta(x)-\frac{\epsilon}{2}\sigma^2(x)(\underline{\phi}_\beta(x))^2 &= \underline{\ell}^\epsilon(\beta)-\kappa(x), \quad x\le\beta,\\
\underline{\phi}_\beta(x) &= -c(x), \quad x\ge\beta,
\end{aligned}
$$

with $\underline{\ell}^\epsilon(\beta) := -b(\beta)c(\beta)+\kappa(\beta)-\frac{1}{2}\sigma^2(\beta)(\epsilon c^2(\beta)+c_x(\beta))$. *Then,* $\phi_\beta(x,\theta)\ge\underline{\phi}_\beta(x)$ *for any* $x\in\mathbb{R}_+$.

Proof. We only prove (1) as the proof of item (2) will be analogous. Let $\gamma<0$ and denote by $\phi^\gamma_\beta(x,\theta)$ the classical solution to (3.10), and set $\phi_\beta(x,\theta) := \phi^0_\beta(x,\theta)$. We define $\psi^\gamma(x) := \phi^\gamma_\beta(x,\theta_2)-\phi_\beta(x,\theta_1)$, so that ψ satisfies

$$
\text{(A.7)}\qquad \begin{cases}
\frac{1}{2}\sigma^2(x)(\psi^\gamma)_x(x)+b(x)\psi^\gamma(x)-\frac{\epsilon}{2}\sigma^2(x)(\psi^\gamma)^2(x)-\epsilon\sigma^2\phi^\gamma_\beta(x,\theta_2)\psi^\gamma(x)\\
\qquad\qquad = \big(\pi(\beta,\theta_2)-\pi(\beta,\theta_1)\big)-\big(\pi(x,\theta_2)-\pi(x,\theta_1)\big)+\gamma, \quad x\in(0,\beta),\\
\psi^\gamma(x)=0, \quad x\in[\beta,\infty).
\end{cases}
$$

For arbitrarily fixed $x\le\beta$, we rewrite the right-hand side of (A.7) as follows

$$
\begin{aligned}
&\big(\ell^\epsilon(\beta,\theta_2)-\ell^\epsilon(\beta,\theta_1)\big)-\big(\pi(x,\theta_2)-\pi(x,\theta_1)\big)+\gamma\\
&\qquad = \big(\pi(\beta,\theta_2)-\pi(\beta,\theta_1)\big)-\big(\pi(x,\theta_2)-\pi(x,\theta_1)\big)+\gamma\\
\text{(A.8)}\qquad &\qquad = \int_{\theta_1}^{\theta_2}\pi_\theta(\beta,\theta)d\theta-\int_{\theta_1}^{\theta_2}\pi_\theta(x,\theta)d\theta+\gamma = \int_x^\beta\int_{\theta_1}^{\theta_2}\pi_{x\theta}(y,\theta)d\theta dy+\gamma,
\end{aligned}
$$

In order to show that $\psi(x) := \psi^0(x)\ge 0$ for any $x\le\beta$ we argue by contradiction. We assume that there exists $x_1 := \sup\{x\in(0,\beta) : \psi^\gamma(x)=0\}$. Then, plugging $x=\beta$ in (A.7) and using the boundary condition $\psi^\gamma(\beta)=0$ we obtain $\frac{1}{2}\sigma^2(\beta)\psi^\gamma_x(\beta)=\gamma$, which implies, $\psi^\gamma_x(\beta)<0$. Therefore, by plugging $x=x_1$ in (A.7) we have

$$
\text{(A.9)}\qquad \frac{1}{2}\sigma^2(x_1)\psi^\gamma_x(x_1)=\int_{x_1}^\beta\int_{\theta_1}^{\theta_2}\pi_{x\theta}(y,\theta)d\theta dy+\gamma<0,
$$

where last inequality is due to Assumption 2.3-(3). This implies $\psi^{\gamma}_x(x_1) < 0$ and contradicts Lemma A.1. Finally, sending $\gamma \to 0^-$, we complete the proof by Lemma 3.1. □

Proposition A.2 (Comparison principle for singularly controlled SDEs)**.** *Let $\mathbb{Q} \in \mathcal{P}(\Omega, \mathcal{F})$, $x_1, x_2 \in \mathbb{R}_+$ and $\theta_1, \theta_2 \in \mathbb{R}_+$ such that $x_1 \le x_2$ and $\theta_1 \le \theta_2$, then the following hold:*

(1) $X_t^{x_1,\xi^*(\theta_1)} \le X_t^{x_1,\xi^*(\theta_2)}$, $\mathbb{Q} \otimes dt-a.s.$;

(2) *let $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}$ be locally Lipschitz functions with $\psi_1(x) \le \psi_2(x)$ for any $x \in \mathbb{R}_+$. Then, if $(X_t^{(i)})_{t\ge 0}$, $i = 1, 2$ are strong solutions to*

$$dX_t^{(i)} = \big(b(X_t^{(i)}) + \psi_i(X_t^{(i)})\sigma(X_t^{(i)})\big)dt + \sigma(X_t^{(i)})dW_t^{\mathbb{Q}} - d\xi_t, \quad i = 1, 2 \tag{A.10}$$

one has that $X_t^{(1)} \le X_t^{(2)}$, $\mathbb{Q} \otimes dt-a.s.$

Proof. The proof follows by combining Proposition 5.2.18 in [48] with Theorem 1.4.1 in [58]. □

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