

MCKEAN-VLASOV EQUATIONS AND NONLINEAR FOKKER-PLANCK EQUATIONS WITH CRITICAL SINGULAR LORENTZ KERNELS

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ABSTRACT. We prove the existence and conditional uniqueness in the Krylov class for SDEs with singular divergence-free drifts in the endpoint critical Lorentz space $L^\infty(0, T; L^{d, \infty}(\mathbb{R}^d))$, $d \geq 2$, which particularly includes the Biot-Savart law. The uniqueness result is shown to be optimal in dimensions $d \geq 3$, by constructing different martingale solutions in the case of supercritical Lorentz drifts. As a consequence, the well-posedness of McKean-Vlasov equations and nonlinear Fokker-Planck equations with critical singular kernels is derived. In particular, this yields the uniqueness of the 2D vorticity Navier-Stokes equations even in certain supercritical-scaling spaces. Furthermore, we prove that the path laws of solutions to McKean-Vlasov equations with critical singular kernel form a nonlinear Markov process in the sense of McKean.

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1. INTRODUCTION AND MAIN RESULTS

Nonlinear Markov evolutions arise in various non-equilibrium statistical models, such as kinetic equations of Vlasov, Boltzmann and Landau. In the seminal paper [39], McKean proposed a deep connection between nonlinear Fokker-Planck equations with a wide class of nonlinear Markov processes. This kind of relationship enables one to study nonlinear parabolic PDEs through underlying stochastic mechanisms governed by SDEs, now referred to as McKean-Vlasov equations, and vice versa.

The purpose of this paper is to make progress, in the spirit of McKean, towards the understanding of McKean-Vlasov equations and nonlinear Fokker-Planck equations, particularly, with divergence-free drifts/*singular kernels* in the *endpoint critical* case for dimensions $d \geq 2$.

One primary model of our nonlinear Fokker-Planck equations is the 2D vorticity Navier-Stokes equation (NSE)

$$\partial_t \rho = \Delta \rho - \operatorname{div}(u\rho), \quad \rho|_{t=0} = \zeta, \quad (\text{NSE})$$

where the velocity field u can be reconstructed from ρ by the Biot-Savart law

$$u(t, x) = \int_{\mathbb{R}^2} K_{\text{BS}}(x - y)\rho(t, dy), \quad K_{\text{BS}}(x) = \frac{1}{2\pi} \frac{(-x_2, x_1)}{x_1^2 + x_2^2}. \quad (1.1)$$

In view of Itô's formula, (NSE) corresponds to the following general McKean-Vlasov equation

$$\begin{cases} dX_t = \int_{\mathbb{R}^d} K(t, X_t - y)\rho(t, dy) dt + \sqrt{2} dW_t \\ \rho(t) = \operatorname{law}(X_t), \quad \rho|_{t=0} = \zeta \in \mathcal{P}(\mathbb{R}^d) \end{cases} \quad (\text{MVE})$$

with $K(t, x) = K_{\text{BS}}(x)$ given by (1.1). The marginal law of McKean-Vlasov solutions satisfies the 2D vorticity (NSE), and thus, (1.1)-(MVE) provide a probabilistic interpretation of (NSE). We note that, rather than in the usual space $L^2_{\text{loc}}(\mathbb{R}^2)$, the Biot-Savart law belongs to the *critical Lorentz space* $L^{2,\infty}(\mathbb{R}^2)$. Hence, the nonlinearity in (NSE) involves a singular kernel. Therefore, achieving the McKean picture for the 2D vorticity NSE, or more general nonlinear Fokker-Planck equations

$$\begin{cases} \partial_t \rho = \Delta \rho - \operatorname{div}((K * \rho)\rho) \\ \rho|_{t=0} = \zeta, \end{cases} \quad (\text{NFPE})$$

where the interaction kernel K can be singular and time-dependent, requires the solvability theory for McKean-Vlasov equations (MVE) with singular kernels. Although the well-posedness of the 2D vorticity NSE has been extensively studied (see, e.g., [16, 18, 19, 21, 24, 5]), the singularity and criticality of kernels pose a major challenge to solve singular McKean-Vlasov equations. As a matter of fact, very few results are known in the case of *critical* singular kernels in dimensions $d \geq 2$.

In contrast to this, the solvability of McKean-Vlasov equations with *subcritical* kernels has been extensively studied, see, e.g., [10, 40, 47, 55]. One efficient approach is to analyze the corresponding linearized version of (MVE), leading to SDEs with subcritical drifts that have been well explored in the literature. We refer to [35, 41, 51, 52] and the references therein.

The well-posedness of SDEs with *critical* drifts is much more challenging. Recently, significant progress has been made by Krylov in a series of papers [30, 32, 33, 34] in dimensions $d \geq 3$, where the strong well-posedness of SDEs with non-endpoint critical Morrey-type drifts was proved. In the endpoint critical case, the case of drifts in the Lorentz space $L_t^\infty L_x^{d,\infty}$ under smallness conditions was addressed in [27] and [34]. For arbitrary divergence-free drifts

in $L_t^\infty L_x^{d,\infty}$ in high dimensions $d \geq 3$, the weak well-posedness of the SDEs was proved by the first and third authors [49].

The remaining 2D case is more subtle than the higher dimensional case. Actually, even in the simple case of SDEs with time-independent drifts, the standard weak sector condition is unclear for the corresponding Dirichlet form in dimension two, while it is valid in dimensions $d \geq 3$. Thus, solving singular SDEs with the *endpoint* critical $L_t^\infty L_x^{2,\infty}$ -Lorentz drifts in dimension two, closely related to the Biot-Savart law, remains a challenging open problem.

In the present paper we address this problem for singular SDEs in dimension two. Furthermore, we achieve the picture of McKean [39] for more general McKean-Vlasov equations and nonlinear Fokker-Planck equations with critical singular Lorentz kernels in all dimensions $d \geq 2$. The main results can be summarized as follows:

- (i) Well-posedness of SDEs with divergence-free drifts in the 2D endpoint critical Lorentz space $L_t^\infty L_x^{2,\infty}$.
- (ii) Well-posedness of McKean-Vlasov equations and nonlinear Fokker-Planck equations with singular kernels in the endpoint critical Lorentz space $L_t^\infty L_x^{d,\infty}$ for dimensions $d \geq 2$. In particular, uniqueness holds in the *Krylov class*.
- (iii) Nonlinear Markov process, in the sense of McKean, formed by the path laws of solutions to McKean-Vlasov equations with critical kernels.

It should be mentioned that uniqueness is in general more difficult than existence of weak solutions to McKean-Vlasov equations and nonlinear Fokker-Planck equations. We refer to [3] for uniqueness in the case of Nemytskii-type nonlinearities. For non-Nemytskii type nonlinearities like the Biot-Savart law in the 2D vorticity NSE, in the recent work [5], which much motivated the present work, uniqueness was obtained in a certain (sub)critical regime, and the nonlinear Markov property of the path laws of solutions to the corresponding McKean-Vlasov equations was proved in the class $L_t^4(L_x^4 \cap L_x^{4/3}) \cap L_t^\infty H_x^{-1}$, in which strong existence and uniqueness is also proved for good initial conditions. In the present work, our general uniqueness results also apply to the 2D vorticity NSE model and render the following new contributions:

- (a) The uniqueness is proved for the 2D vorticity NSE in certain *Krylov class*, which includes a *supercritical* scaling space $L_t^{q'} L_x^{p'}$ with $1/p' + 1/q' > 1$.
- (b) The present proof is of probabilistic nature, which is different from the analytic approach based on the superposition principle in [5]. In particular, it permits to derive the integration representation formula for solutions to (NSE) and associated (MVE) in the Krylov class, with kernels satisfying the Arosón-type estimate, which is important in the derivation of uniqueness.
- (c) The probabilistic proof is also stable under small perturbations of integrable exponents, which reveals that the unique solution class has a non-empty open interior of (p', q') in the supercritical regime.
- (d) The nonlinear Markov property of solutions is derived in the Krylov class $L_t^{q'} L_x^{p'}$ with $1/p' + 1/q' > 1$ for (MVE) associated to the 2D vorticity NSE.

Let us mention that, in view of the Ladyženskaja-Prodi-Serrin (LPS) criteria, weak solutions to NSE in the (sub)critical-scaling regime are unique and automatically in the Leray-Hopf class [12, Theorem 1.3] for $d \geq 2$. In the supercritical-scaling regime, it is usually expected that solutions are not unique. This has been confirmed near one endpoint of the LPS criteria, that is, in the space $C_t L_x^p$ with $p < 2$, for the 2D NSE, recently proved by Cheskidov and Luo [13]. This phenomenon also occurs near another LPS endpoint in the

space $L_t^p L_x^\infty$ with $p < 2$ for the 3D NSE [12], and near two LPS endpoints for the hyperdissipative NSE with viscosity beyond the Lions exponent $5/4$ [36]. The non-uniqueness of weak solutions is also expected in the remaining supercritical regime [12, 36]. Our uniqueness result in the Krylov class reveals that, even in the supercritical regime, there would exist certain unique solution class for the 2D vorticity NSE, as well as for the 2D NSE, see Theorem 1.8 and Corollary 5.2 below.

1.1. Main results.

1.1.1. *SDEs with endpoint critical drifts.* We start with SDEs with endpoint critical drifts, which will serve as the linearization of McKean-Vlasov equations (MVE) when the drift is independent of the marginal law $\rho(t)$.

To be precise, we consider the following SDE in \mathbb{R}^d :

$$X_t^{s,x} = x + \int_s^t b(r, X_r^{s,x}) dr + \sqrt{2}(W_t - W_s), \quad 0 \leq s < t \leq T, \quad (\text{SDE})$$

where b is a divergence-free vector field and belongs to the critical Lorentz space

$$\|b\|_{L_t^\infty L_x^{d,\infty}}^d := \sup_{t \in [0, T]} \sup_{\lambda > 0} \lambda^d |\{x : |b(t, x)| > \lambda\}| < \infty \quad \text{and} \quad \operatorname{div} b = 0. \quad (\text{A}_b)$$

The criticality can be seen from scaling arguments, see Subsubsection 1.2.1 below for detailed explanations.

Let h denote the heat kernel for Δ in \mathbb{R}^d

$$h(t, x) := (4\pi t)^{-\frac{d}{2}} \exp(-|x|^2/4t) \quad (1.2)$$

and

$$\mathcal{I} := \left\{ (p, q) \in (1, \infty)^2 : \frac{d}{p} + \frac{2}{q} < 2 \right\}.$$

The main results for SDEs with endpoint critical drifts in dimension two are formulated in Theorem 1.1 below.

Theorem 1.1 (Well-posedness of 2D critical SDE). *Let $d = 2$. Assume that the drift b satisfies (A_b). Then the following conclusions hold:*

- (i) **Existence:** *For any $0 \leq s < T$ and $x \in \mathbb{R}^2$, there exists a weak solution $(X_t^{s,x})_{t \in [s, T]}$ to (SDE), such that its density $p_{s,t}(x, \cdot)$ satisfies the Aronson-type estimate: for any $0 \leq s \leq t \leq T$ and $x, y \in \mathbb{R}^2$,*

$$\frac{1}{C} h((t-s)/C, x-y) \leq p_{s,t}(x, y) \leq C h(C(t-s), x-y), \quad (\text{AE})$$

where C is a constant only depending on $\|b\|_{L_t^\infty L_x^{2,\infty}}$. In particular, for any $(p, q) \in \mathcal{I}$, the following Krylov-type estimate holds:

$$\mathbf{E} \int_s^T f(t, X_t^{s,x}) dt \leq C \|f\|_{L^q(s, T; L_x^p)}, \quad (\text{KE})$$

where C is a constant depending only on p, q, T and $\|b\|_{L_t^\infty L_x^{2,\infty}}$.

- (ii) **Conditional Uniqueness:** *There exists a pair $(\mathbf{p}, \mathbf{q}) \in \mathcal{I}$, depending only on $\|b\|_{L_t^\infty L_x^{2,\infty}}$, such that the law of the solution to (SDE) is unique within the class of all processes satisfying the following generalized Krylov-type estimate*

$$\mathbf{E} \int_{s+\delta}^T f(t, X_t^{s,x}) dt \leq C_\delta \|f\|_{L^q(s+\delta, T; L_x^p)}, \quad \forall \delta \in (0, T-s), \quad (\text{KE}')$$

where C_δ depends on δ as well as $p, q, T, \|b\|_{L_t^\infty L_x^{2,\infty}}$. Moreover, the collection of distributions $\mathbb{P}_{s,x}$ of $(X^{s,x})$ forms a strong (linear) Markov process.

In the sequel, we say that a process $(X_t)_{t \in [s, T]}$ (or equivalently, a probability measure \mathbb{P} on $C([s, T]; \mathbb{R}^d)$) belongs to the *Krylov class* if X (or the canonical process under \mathbb{P}) satisfies (KE') for some $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{I}$. With slight abuse of terminology, we also say a time-space function $f : (s, T) \times \mathbb{R}^d \rightarrow \mathbb{R}$ belongs to the Krylov class, if $f \in L^{\frac{q}{q-1}}(s + \delta, T; L_x^{\frac{p}{p-1}})$ for any $\delta \in (0, T - s)$ and some $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{I}$.

Let us mention that in the high dimensional case, where $d \geq 3$, the existence of weak solutions and the conditional uniqueness have been proved for (SDE) with the critical $L_t^\infty L_x^{d,\infty}$ -Lorentz drifts by the first and third named authors [49]. The Aronson-type estimate when $d \geq 3$ is implied by the previous work by Qian-Xi [45] and Kinzebulatov-Semënov [28]. As a consequence, together with Theorem 1.1 in the remaining case where $d = 2$, one has the well-posedness and the Aronson-type estimate for all dimensions $d \geq 2$, as formulated in Theorem 1.2 below.

Theorem 1.2. *For any $d \geq 2$, the results in Theorem 1.1 hold with the exponents \mathfrak{p} and \mathfrak{q} potentially depending on d and $\|b\|_{L_t^\infty L_x^{d,\infty}}$.*

1.1.2. *Optimality of well-posedness.* Theorem 1.2 is optimal for dimensions $d \geq 3$ as revealed by the following non-uniqueness result.

Theorem 1.3 (Non-uniqueness of supercritical SDEs). *Let $d \geq 3$. Then, for any $p \in (d/2, d)$, there exists a supercritical divergence-free vector field $b \in L_x^{p,\infty}$, such that there are two distinct weak solutions to (SDE) starting from the origin, and both of them satisfy the Krylov estimate (KE) for any $(p, q) \in \mathcal{I}$.*

As a consequence, we also obtain the non-uniqueness for linear Fokker-Planck equations with supercritical drifts.

Corollary 1.4 (Non-uniqueness of supercritical FPE). *Let $d \geq 3$, and b be the same divergence-free vector field as in Theorem 1.3. Then, there exist at least two distinct solutions to (3.5) in $C_t \mathcal{P}_x$ with initial data δ_0 , where $C_t \mathcal{P}_x$ denotes the set of weakly continuous probability measure-valued curves on $[0, T]$.*

Remark 1.5. (i) *As mentioned before, the well-posedness of (SDE) in dimension two is more difficult than the higher dimensional case. The detailed explanations of their distinctions are contained in Subsubsection 1.2.2 below.*

(ii) *The uniqueness result in Theorem 1.1 is important to establish the uniqueness of solutions to McKean-Vlasov equations and nonlinear Fokker-Planck equations with endpoint critical kernels. In particular, the Krylov class in Theorem 1.1 enables to derive the uniqueness of solutions to the 2D vorticity NSE in certain supercritical spaces. See Theorem 1.8 below.*

(iii) *We also prove the uniqueness for the linear Fokker-Planck equations, see Proposition 3.2 below. We mention that the uniqueness for (SDE) in general cannot imply that of the corresponding linear Fokker-Planck equation. For more details, we refer to the monograph [7], in particular to Section 9.2 which contains examples for non-uniqueness.*

(iv) *It is worth noting that the divergence-free condition is necessary in Theorems 1.2 and 1.3 to guarantee both the existence of weak solutions to SDEs with critical drifts and the uniqueness in the Krylov class. Actually, if the drift b is not divergence free, it was shown in [6, Example 7.4] that (SDE) may not have weak solutions in the case of*

critical drifts in $L^{d,\infty}(\mathbb{R}^d)$, $d \geq 2$. The divergence-free condition of drifts also matches naturally the incompressibility condition of velocity fields in the NSEs.

1.1.3. *McKean-Vlasov equations and nonlinear Fokker-Planck equations with critical kernels.* By virtue of Theorem 1.2, we obtain the well-posedness results for both the McKean-Vlasov equations (MVE) and nonlinear Fokker-Planck equations (NFPE) with critical singular kernels in all dimensions $d \geq 2$.

Let $\mathcal{M}(\mathbb{R}^d)$ denote the space of all real measures on \mathbb{R}^d with finite total variation, and $\mathcal{P}(\mathbb{R}^d)$ the space of all probability measures on \mathbb{R}^d . We denote by $|\zeta|(\mathbb{R}^d)$ the total variation of $\zeta \in \mathcal{M}(\mathbb{R}^d)$. Any $\zeta \in \mathcal{M}(\mathbb{R}^d)$ can be decomposed uniquely as

$$\zeta = \zeta_c + \zeta_a,$$

where ζ_c is the continuous part, i.e., $\zeta_c(\{x\}) = 0$ for all $x \in \mathbb{R}^d$, and $\zeta_a = \sum_i c_i \delta_{x_i}$ is the purely atomic part. Set

$$\mathcal{P}_\varepsilon(\mathbb{R}^d) := \left\{ \zeta \in \mathcal{P}(\mathbb{R}^d) : \zeta_a(\mathbb{R}^d) \leq \varepsilon \right\}.$$

We say a map $\mu : [0, T] \rightarrow \mathcal{M}(\mathbb{R}^d)$ is narrowly continuous if $t \mapsto \int f d\mu_t$ is continuous for all $f \in C_b(\mathbb{R}^d)$.

Theorem 1.6 (Well-posedness of McKean-Vlasov equations). *Let $d \geq 2$. Assume $K \in L_t^\infty L_x^{d,\infty}$ and $\operatorname{div} K = 0$. Then, the following holds:*

- (i) *For any $\zeta \in \mathcal{P}(\mathbb{R}^d)$, there exists at least one weak solution (X_t) to (MVE). Moreover, for each $t \in (0, T]$ the distribution of X_t admits a smooth bounded density.*
- (ii) *There exist $(\mathbf{p}, \mathbf{q}) \in \mathcal{S}$ and $\varepsilon_0 > 0$, depending on d and $\|K\|_{L_t^\infty L_x^{d,\infty}}$, such that if $\zeta \in \mathcal{P}_{\varepsilon_0}(\mathbb{R}^d)$, then the weak solution to equation (MVE) is unique in the class of processes (Y_t) satisfying the following property: for any $\delta \in (0, T)$, there is a constant C_δ such that*

$$\mathbf{E} \int_\delta^T f(t, Y_t) dt \leq C_\delta \|f\|_{L^q(\delta, T; L_x^{\mathbf{p}})}. \quad (1.3)$$

Parallel to the above results of (MVE), we have the well-posedness of (NFPE).

Theorem 1.7 (Well-posedness of nonlinear Fokker-Planck equations). *Let $d \geq 2$. Assume that $K \in L_t^\infty L_x^{d,\infty}$ and $\operatorname{div} K = 0$. Then, the following holds:*

- (i) *For any $\zeta \in \mathcal{M}(\mathbb{R}^d)$, there exists at least one weak solution ρ to (NFPE). Moreover, for each $t \in (0, T]$, $\rho(t)$ is smooth and bounded.*
- (ii) *There exist $(\mathbf{p}, \mathbf{q}) \in \mathcal{S}$ and $\varepsilon_0 > 0$, depending on d and $\|K\|_{L_t^\infty L_x^{d,\infty}}$, such that if $\zeta_a \in \mathcal{M}(\mathbb{R}^d)$ with $|\zeta_a|(\mathbb{R}^d) \leq \varepsilon_0$, then the narrowly continuous weak solution to equation (NFPE) is unique in the class of functions ρ satisfying the following property: for each $\delta \in (0, T)$,*

$$\rho \in L^{\mathbf{q}'}(\delta, T; L_x^{\mathbf{p}'}),$$

where $(\mathbf{p}', \mathbf{q}') = (\mathbf{p}/(\mathbf{p}-1), \mathbf{q}/(\mathbf{q}-1))$.

In the specific case where the singular kernel K is the Biot-Savart law, we have the following uniqueness result for the 2D vorticity NSE in certain scaling-supercritical spaces.

Theorem 1.8 (Well-posedness of 2D vorticity NSE). *For any $\zeta \in \mathcal{M}(\mathbb{R}^2)$, the narrowly continuous weak solution to equation (NSE) is unique in the class of functions ρ satisfying the following property: for each $\delta \in (0, T)$,*

$$\rho \in L^{\mathbf{q}'}(\delta, T; L_x^{\mathbf{p}'}), \quad (1.4)$$

where \mathfrak{p}' and \mathfrak{q}' are the same exponents as in Theorem 1.7 satisfying $1/\mathfrak{p}' + 1/\mathfrak{q}' > 1$.

Remark 1.9. (i) The uniqueness problem of the 2D vorticity NSE has been extensively studied in literature. In the remarkable paper [18], the uniqueness was proved for solutions $w \in C((0, T); L^1 \cap L^\infty)$ satisfying $\|w(t)\|_{L^1} \leq C$ for all $t \in (0, T)$ and $w(t) \rightarrow \mu \in \mathcal{M}(\mathbb{R}^2)$ as $t \rightarrow 0$. In [16], the uniqueness was proved for weak solutions $w \in C([0, T]; \mathcal{D}') \cap L^\infty(0, T; \mathcal{P})$ such that $\nabla w \in L^{2q/(3q-2)}(0, T; L^q)$, $\forall q \in [1, 2)$. Very recently, the uniqueness in the class $L_t^\infty H_x^{-1} \cap L_t^4(L_x^4 \cap L_x^{4/3})$ was proved in [5].

In Theorem 1.7, the uniqueness is derived in the Krylov class for general nonlinear Fokker-Planck equations with critical kernels, which include the 2D vorticity NSE.

(ii) We remark that the Krylov class allows for obtaining uniqueness of solutions to (NSE) in certain scaling-supercritical spaces $L_t^{\mathfrak{q}'} L_x^{\mathfrak{p}'}$ with $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{S}$.

Actually, the 2D vorticity NSE has the invariant scaling

$$\rho_\lambda(t, x) := \lambda^2 \rho(\lambda^2 t, \lambda x), \quad K_\lambda(t, x) := \lambda K(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

Note that, the Biot-Savart kernel is invariant under the scaling, namely, $K_\lambda(t, x) = K(t, x)$, $\forall \lambda > 0$. The above scaling leaves the mixed Lebesgue space $L^q(\mathbb{R}_+; L_x^p)$ invariant if $1/p + 1/q = 1$. For the exponents $(\mathfrak{p}', \mathfrak{q}')$ in (1.4), since $1/\mathfrak{p}' + 1/\mathfrak{q}' > 1$, one has

$$\|\rho_\lambda\|_{L_t^{\mathfrak{q}'} L_x^{\mathfrak{p}'}} = \lambda^{2(1 - \frac{1}{\mathfrak{q}'} - \frac{1}{\mathfrak{p}'})} \rightarrow \infty$$

as $\lambda \rightarrow 0^+$. This justifies to say that $L_t^{\mathfrak{q}'} L_x^{\mathfrak{p}'}$ is a scaling-supercritical space for (NSE).

It is somewhat surprising to find the uniqueness solution class for (NSE) in a supercritical regime, because usually weak solutions exhibit non-uniqueness in the supercritical regime. See, e.g., [8, 9, 12, 13, 36] for NSE in supercritical spaces with respect to the LPS criterion. In contrast to this, Theorem 1.8 reveals that, even in the supercritical regime, there exist certain uniqueness class for the 2D vorticity NSE.

(iii) In Theorems 1.6-1.8, the uniqueness results in the Krylov class for McKean-Vlasov equations and nonlinear Fokker-Planck equations also hold for an open set of the index $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{S}$, thanks to the stability of our probabilistic arguments under small perturbations of the indices in \mathcal{S} .

The present probabilistic proof also avoids the explicit estimates used in [21, (4.19)] and [24, Theorem 6.1] for proving the uniqueness of (fractional) NSE.

1.1.4. *Nonlinear Markov processes.* An important consequence of the above results is the following nonlinear Markov property for the path laws of solutions to (MVE) in the Krylov class. We refer to Section 6 below for the precise definition of nonlinear Markov process.

Theorem 1.10 (Nonlinear Markov process). *Assume that $K \in L_t^\infty L_x^{d, \infty}$ and $\operatorname{div} K = 0$. Then there exists a nonlinear Markov process $\{\mathbb{P}_{s, \zeta}\}_{(s, \zeta) \in [0, T] \times \mathcal{P}(\mathbb{R}^d)}$ in the sense of Definition 6.1, such that for each $(s, \zeta) \in [0, T] \times \mathcal{P}(\mathbb{R}^d)$, the measure $\mathbb{P}_{s, \zeta}$ is the law of a weak solution to (MVE) with the initial condition (s, ζ) , and $\mathbb{P}_{s, \zeta}$ lies in the Krylov class with the same index $(\mathfrak{p}', \mathfrak{q}')$ as in Theorem 1.7. Moreover, the following uniqueness results hold:*

- (i) $\mathbb{P}_{s, \zeta}$, $s \in [0, T]$, $\zeta \in \mathcal{P}_{\varepsilon_0}$, is unique in the Krylov class, where $\varepsilon_0 > 0$ is the small number as in Theorem 1.6.
- (ii) $\mathbb{P}_{s, \zeta}$, $s \in [0, T]$, $\zeta \in \mathcal{P}(\mathbb{R}^2)$, is unique in the Krylov class, if $d = 2$ and K is the Biot-Savart kernel.

1.2. Related literature.

1.2.1. *SDEs with critical and subcritical drifts.* The study of strong well-posedness of SDEs with irregular drifts dates back to the pioneering works by Zvonkin [56] and Veretennikov [51]. The criticality of drifts of SDE can be seen by scaling arguments. More precisely, let $X(t)$ be a solution to (SDE) and $X_\lambda(t) = \lambda^{-1}X(\lambda^2t)$ for $\lambda > 0$. Then,

$$dX_\lambda(t) = b_\lambda(t, X_\lambda(t))dt + \sqrt{2}dW_\lambda(t),$$

where $b_\lambda(t, x) = \lambda b(\lambda^2t, \lambda x)$ and, by the scaling invariance, $W_\lambda(t) = \lambda^{-1}W(\lambda^2t)$ is another standard Brownian motion. One has

$$\|b_\lambda\|_{L_t^q L_x^{p,r}(\tilde{Q}_R)} = \lambda^{1-\frac{d}{p}-\frac{2}{q}} \|b\|_{L_t^q L_x^{p,r}(\tilde{Q}_{\lambda R})}$$

with $\tilde{Q}_R = \{z = (t, y) : s < t < s + R^2, |y - x| < R\}$, which remains invariant if the LPS condition holds

$$\frac{2}{q} + \frac{d}{p} = 1 \text{ with } p, q \in [2, \infty] \text{ and } r \in (0, \infty] \quad (1.5)$$

(see also [6]). For the small-time behavior of the process, it is more reasonable to consider the local norm $\|b_\lambda\|_{L_t^q L_x^{p,r}(\tilde{Q}_R)}$ with $R < \infty$, rather than the global one. Note that $\|b_\lambda\|_{L_t^q L_x^{p,r}(\tilde{Q}_R)} \rightarrow 0$ as $\lambda \rightarrow 0^+$, except the endpoint case $(p, q) = (d, \infty)$. This intuitively indicates that the endpoint critical case is more difficult to analyze than the non-endpoint critical and subcritical cases.

In the literature, there is a large number of results for SDEs with subcritical drifts, see, for instance, [35, 41, 52] and the references therein. One standard strategy in the subcritical case is to construct a homeomorphism, e.g. the Zvonkin transform, to deal with the irregular drifts by solving the corresponding Kolmogorov equations. But it seems not applicable well in the critical case.

Very recently, the progress in dimensions $d \geq 3$ has been obtained by Krylov [31, 32, 33, 34] regarding the existence of strong solutions to SDEs with non-endpoint critical and supercritical drifts. These developments are based on an analytical criterion for the existence of strong solutions initially applied in [31]. Related results obtained via Malliavin calculus for $d \geq 3$ can be found in [27] and [48].

Lastly, we also would like to refer to [17, 23, 25, 37, 54] for SDEs in the supercritical case.

1.2.2. *Distinctions of critical SDEs in dimensions $d = 2$ and $d \geq 3$.* The subtleness in dimension two to solve SDEs with critical $L_x^{d,\infty}$ -Lorentz drifts can be seen as follows.

From the perspective of the theory of Dirichlet forms, the Dirichlet form corresponding to the operator $\Delta + b \cdot \nabla$, where b is time-independent, is given by:

$$\mathcal{E}(u, v) = (\nabla u, \nabla v) - (b \cdot \nabla u, v), \quad \forall u, v \in H^1(\mathbb{R}^d).$$

In dimensions $d \geq 3$, it is a regular Dirichlet form and satisfies the standard sector condition

$$\begin{aligned} \mathcal{E}(u, v) &\leq \sqrt{\mathcal{E}(u, u)}\sqrt{\mathcal{E}(v, v)} + \sqrt{\mathcal{E}(u, u)}\|bv\|_{L^2} \\ &\leq \sqrt{\mathcal{E}(u, u)} \left(\sqrt{\mathcal{E}(v, v)} + C\|b\|_{L^{d,\infty}}\|v\|_{L^{\frac{2d}{d-2}, 2}} \right) \\ &\leq (1 + C\|b\|_{L^{d,\infty}}) \sqrt{\mathcal{E}(u, u)}\sqrt{\mathcal{E}(v, v)}, \quad \forall u, v \in H^1(\mathbb{R}^d), \end{aligned}$$

where the last step is due to estimate (A.6) below. Hence, by virtue of the theory of Dirichlet forms (see [38]), there exists a unique Hunt process associated with the Dirichlet form $(\mathcal{E}, H^1(\mathbb{R}^d))$. But the sector condition is still unclear in the low dimension case $d = 2$.

Another delicate point can be seen from the viewpoint of martingale problems, where a key step is to find a solution to the associated linear backward Kolmogorov equation (2.11) below in the space

$$\mathcal{H}_q^{2,p} := \{u \in L_t^q L_x^p : \partial_t u, \nabla^2 u \in L_t^q L_x^p\} \text{ with } (\mathfrak{p}, \mathfrak{q}) \in \mathcal{I}. \quad (1.6)$$

This requires the integrability $b \cdot \nabla u \in L_t^q L_x^p$ with $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{I}$, which can be achieved in dimensions $d \geq 3$ ([49]), but is more difficult in dimension $d = 2$.

1.3. Ideas of the proof. The main novelties of our proofs can be summarized as follows.

Meyers-type estimate. The keypoint to solve the backward Kolmogorov equation (2.11) in the required space (1.6) is obtaining a high integrability estimate of the gradient $\|\nabla u\|_{L_t^q L_x^{l,1}}$, where $l > d$ and $q > 2l/(l-d)$.

For dimensions $d \geq 3$, this estimate can be derived by proving an a priori Hölder estimate for solutions to (2.11), and utilizing a Gagliardo-Nirenberg type inequality together with the L^q - L^p theory for parabolic equations. See [49] or Figure 3 in Subsection 2.3 below.

For dimension $d = 2$, the primary obstacle is that the L^q - L^p theory gives only a priori estimate of $\|\nabla u\|_{L_t^\nu L_x^\mu}$ with $\mu < 2 = d$, and the standard energy method gives merely an upper bound of $\|\nabla u\|_{L_{t,x}^2}$. These low integrability estimates, however, are insufficient to solve the backward Kolmogorov equation (2.11) in second order Sobolev spaces.

To overcome this limitation, we introduce upgradation procedures to upgrade the integrability of the gradient of solutions. For the convenience of the readers, we present Figure 1 below to illustrate the proof strategy.

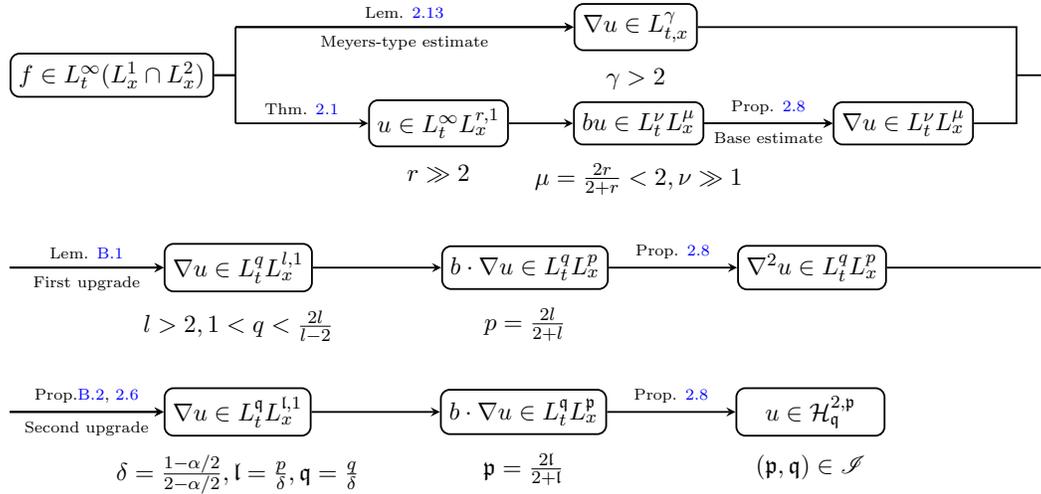


FIGURE 1. Proof Strategy when $d = 2$

We first prove that, in addition to the standard energy bound, the gradient of solutions indeed obeys a *Meyers-type estimate*. That is, the gradient of solutions has an improved integrability in $L_{t,x}^\gamma$ for some $\gamma > 2$, see Lemma 2.13 below. The Meyers-type estimate relies crucially on Gehring's Lemma (Lemma 2.10) and the reverse Hölder estimate (Lemma 2.12), and serves as one of the key steps in upgrading the integrability

$$\|\nabla u\|_{L_t^q L_x^{l,1}} < \infty$$

for certain exponents $l > 2$ and $1 < q < 2l/(l-2)$. It is important that the spatial integrability exponent here can be raised above 2.

Then, in the second upgradation step, we make use of the refined Gagliardo-Nirenberg inequality (see (B.2) below) to further improve the integrability

$$\|\nabla u\|_{L_t^q L_x^l} < \infty,$$

where the exponents $l > 2$ and $q > 2l/(l-2)$. In both steps, it is quite delicate to select appropriate integrability exponents to ensure that the final upgraded integrability exponents $(p, q) \in \mathcal{I}$, i.e., $1/p + 1/q < 1$, as required by the Krylov class.

As a result, we can solve the backward Kolmogorov equation (2.11) in the desired Sobolev space (1.6) in dimension $d = 2$. To the best of our knowledge, the solvability of equation (2.11) in $\mathcal{H}_q^{2,p}$ for dimension two is new in the existing PDE literature.

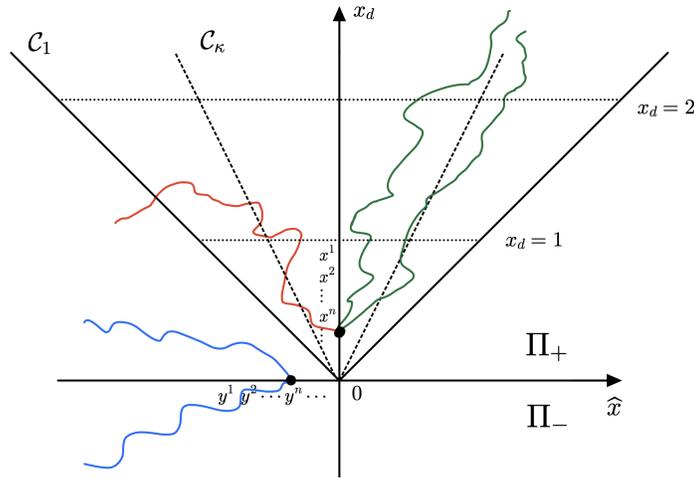


FIGURE 2. Solution paths

The solution process starting from x^n exhibits the green trajectories with high probability.

Construction of non-unique solutions. Concerning the non-uniqueness in Theorem 1.3, the key observation is that the drift in the supercritical regime can be very singular, say, near the origin, so that it suppresses the fluctuations of Brownian motions, thus results in different concentrations of solution paths.

More precisely, we construct a divergence-free vector field $b \in L_x^{p,\infty}$, $p \in (d/2, d)$. It is anti-symmetric with respect to the hyperplane $\Pi = \{x \in \mathbb{R}^d : x_d = 0\}$, and when x is located in the cone $C_\kappa = \{x : x_d > \kappa|\hat{x}|\}$ with $\kappa \geq 1$, it can be very singular in the direction of the last coordinate

$$b(x) \approx (0, \dots, 0, \text{sgn}(x_d)b_d(x)) \approx (0, \dots, 0, \text{sgn}(x_d)|x|^{-d/p}).$$

Then, we consider two sequences $(x^n) \subseteq \{\hat{x} = 0\}$ and $(y^n) \subseteq \Pi$ converging to the singular point 0 in different ways. Because of the anti-symmetry of the drift, the corresponding martingale solutions $\{\mathbb{P}_{y^n}\}$ are invariant under reflection across the hyperplane Π , and so is the limit \mathbb{P}_0 .

In contrast, for the martingale solution \mathbb{P}_{x^n} , the singularity of the drift along the axis x_d forces the solution trajectories starting from the small cone C_κ , with a uniform probability (independent of n), to stay in the larger cone C_1 and return back to C_κ after certain time. As a result, the trajectories are more likely to intersect with the hyperplane $\{x_d = 2\}$ before exiting

\mathcal{C}_1 , and to remain in the region $\{x_d > 1\}$ for a unit time. Intuitively, as illustrated in Figure 2, the solution process follows the green trajectories with high probability. Consequently, the limit $\tilde{\mathbb{P}}_0$ of (\mathbb{P}_{x^n}) is more concentrated on the half space $\Pi_+ = \{x \in \mathbb{R}^d : x_d > 0\}$, and thus, leads to the non-uniqueness $\tilde{\mathbb{P}}_0 \neq \mathbb{P}_0$.

Uniqueness of solutions in the Krylov class. Uniqueness is usually more difficult than existence of weak solutions to McKean-Vlasov equations. See, for instance, the recent work [4] and [5] for an analytic method based on nonlinear Fokker-Planck equations and the superposition principle, and [24, Theorem 6.3] for the well-posedness with initial data in certain Besov spaces including $L^{1+\varepsilon} \cap L^1$ based on the semigroup method.

In the present work, we provide a direct probabilistic method to obtain the uniqueness of solutions in the Krylov class. Building upon our well-posedness results for critical SDEs, we derive an integral representation formula for the solutions in Krylov class to nonlinear Fokker-Planck equations

$$\rho(t, y) = \int_{\mathbb{R}^d} p_{0,t}^{K*\rho}(x, y) \zeta(x) dx,$$

where the kernel $p_{0,t}^{K*\rho}(x, y)$ satisfies the Aronson-type estimate (AE), see (5.6) below. The representation formula is important to close Gronwall-type estimates of solutions in the derivation of the uniqueness.

Concerning the nonlinear Markov property, the one-to-one correspondence for a large class of nonlinear parabolic PDEs and nonlinear Markov processes was proved in [46]. See also [5] and [2] in the case of singular drifts.

Let $\mathcal{Q} \subseteq \mathcal{P}(\mathbb{R}^d)$. In view of the criteria in [46], two conditions are crucial for verifying the nonlinear Markov property for the path laws of solutions to McKean-Vlasov equations:

- Flow property: the marginal distribution $(\mu^{s,\zeta})_{t \in [s, T]}$ satisfies

$$\mu_t^{s,\zeta} \in \mathcal{Q} \text{ and } \mu_t^{s,\zeta} = \mu_t^{r,\mu_r^{s,\zeta}}, \quad \forall s \leq r \leq t, \zeta \in \mathcal{Q} \subseteq \mathcal{P}(\mathbb{R}^d). \quad (1.7)$$

- Extremality: $\mu^{s,\zeta}$ is an extreme point in the convex set of all weakly continuous probability solutions with the initial datum (s, ζ) to the linearized Fokker-Planck equations obtained by freezing $\rho = \mu^{s,\zeta}$ in the convolution term in (NFPE).

Thanks to the uniqueness result of solutions to the nonlinear Fokker-Planck equation, the flow property in the present work is guaranteed. Moreover, our well-posedness result for SDEs ensures that $\mu^{s,\zeta}$ is an extreme point of the aforementioned solution set of the linearized Fokker-Planck equation. As a consequence, we obtain the nonlinear Markov property for (MVE).

Organization: This paper is organized as follows: Section 2 is dedicated to establishing the solvability of the linear backward Kolmogorov equation (2.11) in certain second order Sobolev spaces when b satisfies (A_b). The proofs of Theorem 1.1 and Theorem 1.6 are presented in Section 3 and Section 5, respectively. In Section 4, we provide an example demonstrating the optimality of the condition (A_b). At last, in Appendix A we present the preliminaries of Lorentz spaces. Then, Appendix B contains several useful interpolation estimates in Lorentz spaces used in this paper.

Notations. Let T be any fixed time horizon. For $p \in [1, \infty]$, p' denotes its conjugate number $\frac{p}{p-1}$. For $z = (t, x) \in \mathbb{R} \times \mathbb{R}^d$, we set $Q_R(z) := (t - R^2, t) \times B_R(x)$ and $Q_R := (-R^2, 0) \times B_R$. For any $p, q \in (1, \infty)$, let

$$\|f\|_{L_t^q L_x^p} := \|f\|_{L^q([0, T]; L^p(\mathbb{R}^d))}.$$

Moreover, let $\|f\|_{\dot{H}_x^{s,p}} := \|\Lambda^s f\|_{L_x^p}$ and $\|f\|_{H_x^{s,p}} := \|f\|_{L_x^p} + \|\Lambda^s f\|_{L_x^p}$, where $\Lambda := (-\Delta)^{\frac{1}{2}}$.

2. LINEAR BACKWARD KOLMOGOROV EQUATIONS

In this section, we study the solvability of the backward Kolmogorov equation corresponding to (SDE) when the drift b satisfies (A_b) . We mainly focus on the most delicate case where $d = 2$. The method also applies to high dimensions $d \geq 3$.

Define the Kolmogorov operator by

$$A_t u := \Delta u + b(t, \cdot) \cdot \nabla u,$$

and the dual Fokker-Planck operator

$$A_t^* u := \Delta u - \operatorname{div}(b(t, \cdot)u) = \Delta u - b(t, \cdot) \cdot \nabla u.$$

To avoid inessential issues arising from the singularity of b , in the rest of this section, we assume that $b \in C_b^\infty(\mathbb{R}^{d+1})$ and b is divergence-free. We note that the constants in all of the following a priori estimates do not depend on the regularity of b , but only on the critical Lorentz-norm $\|b\|_{L_t^\infty L_x^{d,\infty}}$, so the results in this section are valid for the drift b satisfying (A_b) .

2.1. Aronson-type estimate. Let us first prove the Aronson-type estimate for the fundamental solutions of $\partial_t - A_t^*$.

Theorem 2.1 (Aronson-type estimate). *Let $d \geq 2$. Then, $\partial_t - A_t^*$ admits a fundamental solution $p_{s,t}(x, y)$, $t \in [s, T]$, $x, y \in \mathbb{R}^d$, i.e.*

$$\begin{cases} \partial_t p_{s,t}(x, y) = [A_t^* p_{s,t}(x, \cdot)](y) \\ \lim_{t \downarrow s} p_{s,t}(x, y) = \delta(x - y), \end{cases}$$

and $p_{s,t}(x, y)$ satisfies

$$\int_{\mathbb{R}^d} p_{s,r}(x, z) p_{r,t}(z, y) dz = p_{s,t}(x, y), \quad \int_{\mathbb{R}^d} p_{s,t}(x, y) dy = 1,$$

and

$$\frac{1}{C} h((t-s)/C, x-y) \leq p_{s,t}(x, y) \leq C h(C(t-s), x-y), \quad (2.1)$$

where h is the heat kernel for Δ on \mathbb{R}^d given by (1.2), C depends only on the dimension d and $\|b\|_{L_t^\infty L_x^{d,\infty}}$. Moreover, there exist constants $\alpha \in (0, 1)$ and $C > 0$, only depending on d and $\|b\|_{L_t^\infty L_x^{d,\infty}}$, such that for any $t > s$ and $x, y, y' \in \mathbb{R}^d$,

$$\begin{aligned} & |p_{s,t}(x, y) - p_{s,t}(x, y')| \\ & \leq C \left[\left(\frac{|y - y'|}{\sqrt{t-s}} \right)^\alpha \wedge 1 \right] [h(C(t-s), x-y) + h(C(t-s), x-y')]. \end{aligned} \quad (2.2)$$

Remark 2.2. (i) The case where $d \geq 3$ follows from [28]. For the convenience of readers, we give a unified estimate for both the cases $d = 2$ and $d \geq 3$.

(ii) Since $p_{s,t}(x, y)$ also satisfies the backward equation

$$\begin{cases} \partial_s p_{s,t}(x, y) = [A_s p_{s,t}(\cdot, y)](x) \\ \lim_{s \uparrow t} p_{s,t}(x, y) = \delta(x - y), \end{cases}$$

and the operators A_t and A_t^* have the same form, except the opposite signs of the drift coefficients, we also have

$$\begin{aligned} & |p_{s,t}(x, y) - p_{s,t}(x', y)| \\ & \leq C \left[\left(\frac{|x - x'|}{\sqrt{t - s}} \right)^\alpha \wedge 1 \right] [h(C(t - s), x - y) + h(C(t - s), x' - y)]. \end{aligned} \quad (2.3)$$

(iii) In light of Theorem 2.1, using standard approximation arguments one also has that for any drift b satisfying (A_b), there exists a Markov process $(\mathbb{P}_{s,x}, X_t)$ such that

$$\mathbb{P}_{s,x}(X_s = x) = 1, \quad \mathbb{P}_{s,x}(X_t \in A) = \int_A p_{s,t}(x, y) dy,$$

where $p_{s,t}$ is the fundamental solution as in Theorem 2.1.

The proof of Theorem 2.1 follows from the Nash iteration as in [42, 15, 45]. Two key ingredients are Nash's inequality

$$\|u\|_{L_x^2}^{2+\frac{4}{d}} \leq C_d \|\nabla u\|_{L_x^2}^2 \|u\|_{L_x^1}^{\frac{4}{d}}, \quad \forall u \in L^1 \cap \dot{H}^1, \quad (2.4)$$

and the following lemma.

Lemma 2.3 ([15]). *Suppose that w is a nonnegative, nondecreasing continuous function on $[0, \infty)$. Let $p \geq 2, a, \beta, \Gamma > 0$ and $\delta \in (0, 1]$. Then, there exists C depending only on a and β such that if*

$$u'(t) \leq -\frac{a t^{p-2} (u(t))^{1+\beta p}}{p (w(t))^{\beta p}} + p\Gamma u(t),$$

then

$$t^{\frac{1}{\beta} - \frac{1}{\beta p}} u(t) \leq \left(\frac{Cp^2}{\delta} \right)^{\frac{1}{\beta p}} e^{\frac{\Gamma \delta t}{p}} w(t).$$

Proof of Theorem 2.1. (i) **Upper bound:** As in [15], for any $\alpha \in \mathbb{R}^d$, we define

$$\varphi(x) = \alpha \cdot x, \quad \phi = e^{-\varphi}, \quad A_t^{\varphi*} f = \phi^{-1} A_t^*(f\phi),$$

and

$$p_{s,t}^{\varphi}(x, y) = \phi^{-1}(y) p_{s,t}(x, y) \phi(x), \quad P_{s,t}^{\varphi*} f := \int_{\mathbb{R}^d} p_{s,t}^{\varphi}(x, \cdot) f(x) dx.$$

By definition, $p_{s,t}^{\varphi}$ is the fundamental solution of $\partial_t - A_t^{\varphi*}$, and

$$\begin{aligned} A_t^{\varphi*} f &= \phi^{-1} \Delta(f\phi) - \phi^{-1} b \cdot \nabla(f\phi) \\ &= \Delta f + 2\nabla f \cdot \nabla \log(\phi) + f \frac{\Delta \phi}{\phi} - b \cdot \nabla f - b \cdot \nabla \log(\phi) f \\ &= \Delta f - (b + 2\alpha) \cdot \nabla f + (|\alpha|^2 + \alpha \cdot b) f. \end{aligned} \quad (2.5)$$

For any $f \in L_x^1 \cap L_x^\infty$, put $f_t = P_{0,t}^{\varphi^*} f$. Then $\partial_t f_t = A_t^{\varphi^*} f_t$. Using (B.6), (2.4) and (2.5), we get that for any $p \in [1, \infty)$,

$$\begin{aligned}
\frac{d}{dt} \int f_t^{2p} &= 2p \int f_t^{2p-1} \partial_t f_t = 2p \int f_t^{2p-1} A_t^{\varphi^*} f_t \\
&= - (4 - 2/p) \int |\nabla f_t^p|^2 + 2p \int |\alpha|^2 f_t^{2p} + 2p \int \alpha \cdot b f_t^{2p} \\
&\leq - (4 - 2/p) \|\nabla f_t^p\|_{L_x^2}^2 + p|\alpha|^2 \|f_t^p\|_{L_x^2}^2 + C p |\alpha| \|b\|_{L_t^\infty L_x^{d,\infty}} \|f_t^p\|_{L_x^2} \|\nabla f_t^p\|_{L_x^2} \\
&\leq - \|\nabla f_t^p\|_{L_x^2}^2 + \left(1 + C \|b\|_{L_t^\infty L_x^{d,\infty}}^2\right) p^2 |\alpha|^2 \|f_t^p\|_{L_x^2}^2 \\
&\leq - c \frac{\|f_t^p\|_{L_x^2}^{2+\frac{4}{d}}}{\|f_t^p\|_{L_x^1}^{\frac{4}{d}}} + \left(1 + C \|b\|_{L_t^\infty L_x^{d,\infty}}^2\right) p^2 |\alpha|^2 \|f_t^p\|_{L_x^2}^2,
\end{aligned} \tag{2.6}$$

where c and C only depend on d .

Let

$$u_p(t) := \|f_t\|_{L_x^{2p}}, \quad w_p(t) := \sup_{0 \leq s \leq t} s^{\frac{d}{4} - \frac{d}{2p}} \|f_s\|_{L_x^p}.$$

Then, (2.6) implies

$$u_p'(t) \leq -\frac{c}{p} \frac{t^{p-2} (u_p(t))^{1+4p/d}}{(w_p(t))^{4p/d}} + C \left(1 + \|b\|_{L_t^\infty L_x^{d,\infty}}^2\right) p |\alpha|^2 u_p(t).$$

Applying Lemma 2.3 we get

$$\begin{aligned}
\|f_t\|_\infty &\leq \limsup_{k \rightarrow \infty} w_{2^k}(t) \leq C(\varepsilon) \exp(\varepsilon |\alpha|^2 t) w_2(t) \\
&\leq C(\varepsilon) \exp(\varepsilon |\alpha|^2 t) \sup_{s \in [0, t]} \|f_s\|_{L_x^2}.
\end{aligned}$$

Using (2.6) again with $p = 1$ we derive that

$$\|f_t\|_{L_x^2}^2 \leq \exp\left(C \left(1 + \|b\|_{L_t^\infty L_x^{d,\infty}}^2\right) |\alpha|^2 t\right) \|f\|_{L_x^2}^2.$$

Combining the above two estimates together one sees

$$\|P_{0,t}^{\varphi^*}\|_{L_x^2 \rightarrow L_x^\infty} \leq C \exp\left(C \left(1 + \|b\|_{L_t^\infty L_x^{d,\infty}}^2\right) |\alpha|^2 t\right).$$

Noting that b is divergence-free, by duality, we also have

$$\|P_{t,2t}^{\varphi^*}\|_{L_x^1 \rightarrow L_x^2} \leq C \exp\left(C \left(1 + \|b\|_{L_t^\infty L_x^{d,\infty}}^2\right) |\alpha|^2 t\right).$$

Therefore,

$$\|P_{0,t}^{\varphi^*}\|_{L_x^1 \rightarrow L_x^\infty} \leq C \exp\left(C \left(1 + \|b\|_{L_t^\infty L_x^{d,\infty}}^2\right) |\alpha|^2 t\right),$$

i.e.,

$$p_{0,t}(x, y) \leq C \exp\left(C \left(1 + \|b\|_{L_t^\infty L_x^{d,\infty}}^2\right) |\alpha|^2 t + \alpha \cdot (y - x)\right).$$

Letting $\alpha = (y - x)(2C(1 + \|b\|_{L_t^\infty L_x^{d,\infty}}^2)t)^{-1}$, we then obtain the upper bound estimate.

(ii) **Lower bound:** Regarding the lower bound, following [15], we set

$$\mu(x) := (2\pi)^{-\frac{d}{2}} \exp(-|x|^2/2)$$

and

$$G(t, y) := \int_{\mathbb{R}^d} \log p_{1-t,1}(x, y) \mu(x) dx, \quad t \in (0, 1] \text{ and } y \in \mathbb{R}^d.$$

By a straightforward computation with the integration-by-parts formula, we obtain

$$\begin{aligned} \partial_t G(t, y) &= \int_{\mathbb{R}^d} [\Delta_x p_{1-t,1}(x, y) + b(1-t, x) \cdot \nabla_x p_{1-t,1}(x, y)] \frac{\mu(x)}{p_{1-t,1}(x, y)} dx \\ &= \int_{\mathbb{R}^d} [x \cdot \nabla_x \log p_{1-t,1}(x, y) + |\nabla_x \log p_{1-t,1}(x, y)|^2] \mu(x) dx \\ &\quad + \int_{\mathbb{R}^d} b(1-t, x) \cdot [\nabla \mu(x) \log p_{1-t,1}(x, y)] dx \\ &\geq -C \|\nabla_x \log p_{1-t,1}(\cdot, y)\|_{L^2(\mu)} + \|\nabla_x \log p_{1-t,1}(\cdot, y)\|_{L^2(\mu)}^2 \\ &\quad - C \|b\|_{L_t^\infty L_x^d} \|\nabla \mu \log p_{1-t,1}(\cdot, y)\|_{L_x^{d',1}} \\ &\geq -C - C \|\cdot \log p_{1-t,1}(\cdot, y) \mu\|_{L_x^{d',1}} + \frac{3}{4} \|\nabla_x \log p_{1-t,1}(\cdot, y)\|_{L^2(\mu)}^2. \end{aligned}$$

In the case where $d = 2$, we have by (B.6),

$$\begin{aligned} \|\cdot \log p_{1-t,1}(\cdot, y) \mu\|_{L_x^{d',1}} &\leq C \|\cdot \mu^{\frac{1}{3}}\|_{L_x^{4,2}} \|f \mu^{\frac{2}{3}}\|_{L_x^{4,2}} \\ &\leq C \|f \mu^{\frac{2}{3}}\|_{L_x^2}^{\frac{1}{2}} \|\nabla(f \mu^{\frac{2}{3}})\|_{L_x^2}^{\frac{1}{2}} \\ &\leq C_\varepsilon \|f \mu^{\frac{2}{3}}\|_{L_x^2} + \varepsilon \|f \cdot \mu^{\frac{2}{3}}\|_{L_x^2} + \varepsilon \|\nabla f \mu^{\frac{2}{3}}\|_{L_x^2} \\ &\leq C_\varepsilon \|f\|_{L^2(\mu)} + \varepsilon \|\nabla f\|_{L^2(\mu)}^2; \end{aligned}$$

While for $d \geq 3$, Hölder's inequality (A.2) yields

$$\|\cdot \log p_{1-t,1}(\cdot, y) \mu\|_{L_x^{d',1}} \leq C \|\cdot \sqrt{\mu}\|_{L_x^{\frac{2d}{d-2},2}} \|f \sqrt{\mu}\|_{L_x^2} \leq C \|f\|_{L^2(\mu)}.$$

Thus, we conclude that for any $d \geq 2$,

$$\partial_t G(t, y) \geq \frac{1}{2} \|\nabla_x \log p_{1-t,1}(\cdot, y)\|_{L^2(\mu)}^2 - C \|\log p_{1-t,1}(\cdot, y)\|_{L^2(\mu)} - C. \quad (2.7)$$

Then the proof for the lower bound can be argued in an analogous way as in [15] and [45], so the details are omitted here.

(iii) **Hölder regularity estimate.** We note that the Aronson-type estimate implies Nash's continuity theorem (see, e.g., [42]): suppose that u is a solution to equation

$$\partial_t u = A_t^* u \quad (\text{or } \partial_t u = A_t u)$$

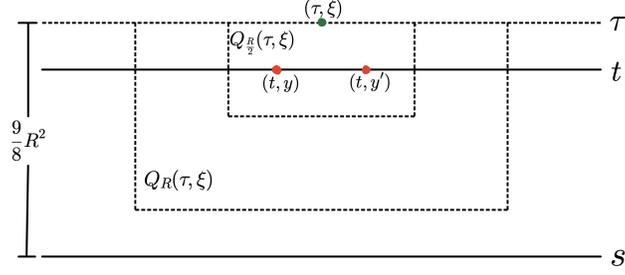
in $Q_R(\tau, \xi)$, then there exist two universal constants $\alpha \in (0, 1)$ and $C > 0$ such that

$$|u(t, y) - u(t', y')| \leq C \left(\frac{|t - t'|^{1/2} + |y - y'|}{R} \right)^\alpha \left(\sup_{Q_R(\tau, \xi)} u - \inf_{Q_R(\tau, \xi)} u \right) \quad (2.8)$$

for all $(t, y), (t', y') \in Q_{R/2}(\tau, \xi)$.

Below we set that for any $0 \leq s < t < \infty$,

$$R := \sqrt{t - s} \quad \text{and} \quad \tau := t + \frac{R^2}{8} = s + \frac{9R^2}{8}.$$

FIGURE 3. Cube $Q_R(\tau, \xi)$

We consider two cases where $|y - y'| \leq R/2$ and $|y - y'| > R/2$, respectively.

(iii.a) Suppose that $y, y' \in \mathbb{R}^d$ satisfying $|y - y'| \leq R/2$. Let $\xi = \frac{y+y'}{2}$. Applying (2.8) to $u(t, y) = p_{s,t}(x, y)$, we obtain that

$$\begin{aligned} |p_{s,t}(x, y) - p_{s,t}(x, y')| &\leq C \left(\frac{|y - y'|}{R} \right)^\alpha \sup_{\substack{z \in B_R(\xi); \\ r \in (\tau - R^2, \tau)}} p_{s,r}(x, z) \\ &\leq C |y - y'|^\alpha (t - s)^{-\frac{d+\alpha}{2}} \sup_{\substack{z \in B_R(\xi); \\ r \in (s + R^2/8, s + (9R^2/8))}} \exp\left(\frac{-|x - z|^2}{CR^2} \right). \end{aligned}$$

Since

$$\sup_{z \in B_R(\xi)} \exp\left(\frac{-|x - z|^2}{CR^2} \right) \leq 1 \leq C \exp\left(\frac{-|x - y|^2}{CR^2} \right) \quad \text{when } x \in B_{2R}(\xi)$$

and

$$\sup_{z \in B_R(\xi)} \exp\left(\frac{-|x - z|^2}{CR^2} \right) \asymp \exp\left(\frac{-|x - y|^2}{C'R^2} \right) \quad \text{when } x \in B_{2R}^c(\bar{y}),$$

we get

$$|p_{s,t}(x, y) - p_{s,t}(x, y')| \leq C |y - y'|^\alpha (t - s)^{-\frac{\alpha}{2}} h(C(t - s), x - y). \quad (2.9)$$

(iii.b) If $|y - y'| > R/2$, then we have

$$\begin{aligned} |p_{s,t}(x, y) - p_{s,t}(x, y')| &\leq |p_{s,t}(x, y)| + |p_{s,t}(x, y')| \\ &\leq h(C(t - s), x - y) + h(C(t - s), x - y'). \end{aligned} \quad (2.10)$$

Combining (2.9) and (2.10) together we obtain the desired conclusion. \square

2.2. Hölder regularity estimate. Now, let us consider the Kolmogorov equation

$$\partial_t u = A_t u + cu + f, \quad u(0) = 0 \quad (2.11)$$

or

$$\partial_s v + A_s v + cv = f, \quad v(T) = 0. \quad (2.12)$$

The main result of this subsection is Proposition 2.6 below concerning the Hölder regularity estimate of solutions to the Kolmogorov equation.

Definition 2.1. Let I be an open interval in \mathbb{R}_+ and D a domain of \mathbb{R}^d . Set $Q := I \times D$. We say that $u \in L_t^\infty L_x^2(Q) \cap L_t^2 H_x^1(Q)$ is a subsolution (resp. supersolution) of

$$\partial_t u = A_t u + cu + f \quad (2.13)$$

in Q , if for any $\varphi \in C_c^\infty(Q)$ with $\varphi \geq 0$,

$$\int_Q [-u \partial_t \varphi + \nabla u \cdot \nabla \varphi + (b \cdot \nabla u) \varphi - cu \varphi] \leq (\text{resp. } \geq) \int_Q f \varphi.$$

If $u \in C_t L_x^2 \cap L_t^2 H_x^1$ is a solution to (2.13) on $(0, T) \times \mathbb{R}^d$ and $u(0) = 0$, then we say that u is a solution to (2.11).

We first have the following energy estimate, the proof is standard and thus is omitted.

Proposition 2.4 (Energy estimate). *Assume that $f = g + \operatorname{div} F$ with $g \in L_t^1 L_x^2$ and $F \in L^2(\mathbb{R}^d; \mathbb{R}^d)$. Let $u \in C_t L_x^2 \cap L_t^2 H_x^1$ be the solution to (2.11). Then*

$$\|u\|_{L_t^\infty L_x^2} + \|\nabla u\|_{L_{t,x}^2} \leq C \left(\|g\|_{L_t^1 L_x^2} + \|F\|_{L_{t,x}^2} \right),$$

where C only depends on $\|c\|_{L^\infty}$ and T . When $d = 2$ and $c \equiv 0$, one has

$$\|u\|_{L_t^\infty L_x^2}^2 + \|\nabla u\|_{L_{t,x}^2}^2 \leq C_1 \|f\|_{L_{t,x}^{\frac{4}{3}}}^2,$$

where C_1 is independent of T .

The following fact will be used frequently in this paper.

Lemma 2.5. *Let h be the heat kernel given by (1.2). Then, for any $l, \alpha \geq 1$, it holds that*

$$\|h(t, \cdot)\|_{L_x^{l,\alpha}} \leq C t^{-\frac{d}{2}(1-\frac{1}{l})}. \quad (2.14)$$

As a consequence, for any $(p, q) \in \mathcal{I}$, $r \geq p$ and $\beta \geq 1$,

$$\sup_{t \in [0, T]} \left\| \int_0^t f(t-s) * h(s) ds \right\|_{L_x^{r,\beta}} \leq C \|f\|_{L_t^q L_x^p}. \quad (2.15)$$

Proof. Since $h(t, x) = t^{-\frac{d}{2}} h(1, x/\sqrt{t})$, one has

$$\|h(t, \cdot)\|_{L_x^{l,\alpha}} = t^{-\frac{d}{2}(1-\frac{1}{l})} \|h(1, \cdot)\|_{L_x^{l,\alpha}},$$

from which (2.14) can be derived.

Then, an application of Proposition A.1 (iii) and (2.14) yields

$$\sup_{t \in [0, T]} \left\| \int_0^t f(t-s) * h(s) ds \right\|_{L_x^{r,\beta}} \leq C \sup_{t \in [0, T]} \int_0^t s^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{r})} \|f(t-s, \cdot)\|_{L_x^p} ds \leq C \|f\|_{L_t^q L_x^p}.$$

where we also used the fact that $-\frac{d}{2}(\frac{1}{p} - \frac{1}{r})(1 - \frac{1}{q})^{-1} > -1$ in the last step, due to the conditions on p, q and r . Thus, (2.15) is proved. \square

Proposition 2.6 (Hölder regularity estimate). *Let $d \geq 2$ and $(p, q) \in \mathcal{I}$. Then, for any $f \in L_t^q L_x^p$, the Kolmogorov equation (2.11) admits a weak solution u such that*

$$\sup_{t \in [0, T]} \|u(t)\|_{C_x^\alpha} \leq C \|f\|_{L_t^q L_x^p}, \quad (2.16)$$

where α depends only on d, p, q and $\|b\|_{L_t^\infty L_x^{d,\infty}}$, and C depends on d, α, p, q, T , $\|b\|_{L_t^\infty L_x^{d,\infty}}$ and $\|c\|_{L_{t,x}^\infty}$.

Proof. By standard approximation arguments, it suffices to consider the smooth case where $b, c, f \in C_b^\infty(\mathbb{R}^d)$ and to prove that

$$\|v(t)\|_{C_x^\alpha} \leq C \|f\|_{L_t^q L_x^p}, \quad t \in [0, T],$$

for v satisfying (2.12).

For this purpose, by the Feynman-Kac formula, we derive

$$v(s, x) = \mathbb{E}_{s,x} \int_s^T f(t, X_t) \exp\left(-\int_s^t c(r, X_r) dr\right) dt,$$

where $(\mathbb{P}_{s,x}, X_t)$ is the Markov process corresponding to A_t . Then, the Aronson-type estimate (2.1) yields

$$|v(s, x)| \leq C e^{\|c\|_{L^\infty} T} \int_s^T \int_{\mathbb{R}^d} |f(t, y)| h(C(t-s), x-y) dy dt.$$

Let $l = p/(p-1)$ in (2.14). Since $d/p + 2/q < 2$, we get

$$\|v(s, \cdot)\|_{L_x^\infty} \leq C \int_0^{T-s} t^{-\frac{d}{2p}} \|f(s+t, \cdot)\|_{L^p} dt \leq C \|f\|_{L_t^q L_x^p}. \quad (2.17)$$

Similarly, one can also derive that

$$\|v\|_{L_t^q L_x^p} \leq C \|f\|_{L_t^q L_x^p}. \quad (2.18)$$

Now, let $g := cv + f$. Since

$$\partial_s v + A_s v + g = 0, \quad v(T) = 0,$$

it holds that

$$v(s, x) = \mathbb{E}_{s,x} \int_s^T g(t, X_t) dt = \int_s^T \int_{\mathbb{R}^d} p_{s,t}(x, y) g(t, y) dy dt.$$

Let $R = |x - x'| \ll 1$. By (2.3) and Hölder's inequality, there exists $\alpha \in (0, 1)$ such that

$$\begin{aligned} |v(s, x) - v(s, x')| &\leq \int_s^T |g(t, y)| |p_{s,t}(x, y) - p_{s,t}(x', y)| dt \\ &\leq C \int_s^{s+R^2} |g(t, y)| (p_{s,t}(x, y) + p_{s,t}(x', y)) dy dt \\ &\quad + CR^\alpha \int_{s+R^2}^T (t-s)^{-\frac{\alpha}{2}} dt \int_{\mathbb{R}^d} |g(t, y)| (t-s)^{-\frac{d}{2}} \exp\left(\frac{-|x-y|^2}{C(t-s)}\right) dy \\ &\leq C \int_0^{R^2} t^{-\frac{d}{2p}} \|g(s+t, \cdot)\|_{L_x^p} dt + CR^\alpha \int_{R^2}^T t^{-\frac{\alpha}{2} - \frac{d}{2p}} \|g(s+t, \cdot)\|_{L_x^p} dt \\ &\leq CR^{2-\frac{d}{p}-\frac{2}{q}} \|g\|_{L^q(s, T; L_x^p)} + CR^\alpha \|g\|_{L^q(s, T; L_x^p)} \left(\int_{R^2}^T t^{-\frac{\alpha q'}{2} - \frac{dq'}{2p}} dt\right)^{\frac{1}{q'}}. \end{aligned}$$

Re-selecting the parameter $\alpha \in (0, 2 - d/p - 2/q)$ and combining (2.18) with the above estimate we thus obtain

$$\begin{aligned} |v(s, x) - v(s, x')| &\leq C |x - x'|^\alpha \|g\|_{L^q(0, T; L_x^p)} \\ &\leq C |x - x'|^\alpha \left(\|v\|_{L_t^q L_x^p} + \|f\|_{L_t^q L_x^p}\right) \\ &\leq C |x - x'|^\alpha \|f\|_{L_t^q L_x^p}. \end{aligned}$$

This together with (2.17) prove the desired assertion. \square

2.3. Sobolev regularity estimate. The main result of this subsection is the following second order Sobolev regularity estimate for backward Kolmogorov equations, which is crucial in the proof of Theorem 1.1.

Theorem 2.7 (Second order Sobolev regularity estimate). *Let $d \geq 2$. Assume that b satisfies (A_b) . Then, there exists $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{S}$, which only depends on d and $\|b\|_{L_t^\infty L_x^{d,\infty}}$, such that for any $c \in L_t^\infty(L_x^1 \cap L_x^\infty)$ and $f \in L_t^\infty(L_x^1 \cap L_x^d)$, the weak solution u to (2.11) satisfies*

$$\|u\|_{L_t^q W_x^{2,\mathfrak{p}}} + \|\partial_t u\|_{L_t^q L_x^\mathfrak{p}} \leq C \|f\|_{L_t^\infty(L_x^1 \cap L_x^d)}, \quad (2.19)$$

where C only depends on $d, \mathfrak{p}, \mathfrak{q}, T, \|b\|_{L_t^\infty L_x^{d,\infty}}$ and $\|c\|_{L_t^\infty(L_x^1 \cap L_x^\infty)}$.

In order to prove Theorem 2.7, let us first recall the following L^q - L^p estimate for the heat equation proved by Krylov [29].

Proposition 2.8 ([29]). *Let $p, q \in (1, \infty)$, $\alpha \in \mathbb{R}$. Assume that $f \in L_t^q H_x^{\alpha,p}$. Then, the heat equation*

$$\partial_t u - \Delta u = f \quad \text{in } (0, T) \times \mathbb{R}^d, \quad u(0) = 0, \quad (2.20)$$

has a unique solution in $L_t^q H_x^{2+\alpha,p}$, which satisfies the estimate

$$\|\partial_t u\|_{L_t^q H_x^{\alpha,p}} + \|u\|_{L_t^q H_x^{2+\alpha,p}} \leq C \|f\|_{L_t^q H_x^{\alpha,p}}, \quad (2.21)$$

where $C = C(d, p, q, T)$.

The next result gives a parabolic version of Sobolev's embedding (see, e.g., [49]).

Proposition 2.9 (Parabolic-type Sobolev embedding). *Let $p, q \in (1, \infty)$, $r \in [p, \infty)$ and $s \in [q, \infty)$. Assume $\partial_t u \in L_t^q L_x^p$, $u \in L_t^q W_x^{2,p}$. If $1 < 2/q + d/p = 2/s + d/r + 1$, then*

$$\|\nabla u\|_{L_t^s L_x^r} \leq C \left(\|\partial_t u\|_{L_t^q L_x^p} + \|\nabla^2 u\|_{L_t^q L_x^p} \right), \quad (2.22)$$

where C only depends on d, p, q, r, s .

The case where $d \geq 3$: The proof is essentially presented in [49], see Figure 4 for the strategy of the proof. Hence we omit the details here, but remark that the strategy presented in Figure 4 can be implemented by selecting the parameters as

$$r > \frac{d}{1/\delta - 2}, \quad q > \frac{2}{1/\delta - d/r - 2}$$

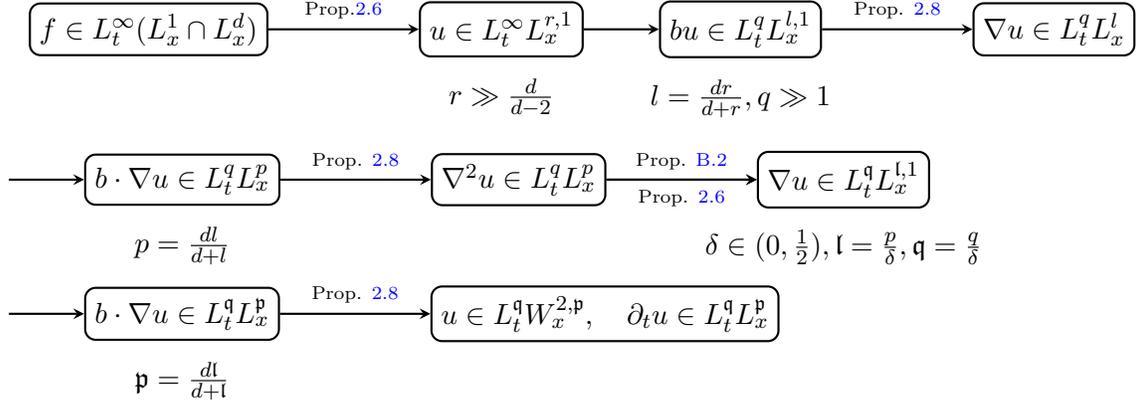
such that

$$\frac{d}{\mathfrak{p}} + \frac{2}{\mathfrak{q}} = 1 + 2\delta + \delta \left(\frac{d}{r} + \frac{2}{q} \right) < 2.$$

The case where $d = 2$: As mentioned before, the 2D case is much more delicate than the high dimensional case $d \geq 3$. One major obstacle is that the above strategy can not establish $\nabla u \in L_t^q L_x^l$ with $l > 2 = d$. It thus limits the application of Proposition 2.8 to implement the above strategy.

To overcome this problem, we introduce the upgradation procedure as shown in Figure 1 of Section 1, based on a new Meyers-type estimate and the refined Gagliardo-Nirenberg estimate. In order to implement the upgradation strategy, let us first recall the famous Gerhing's Lemma. Let

$$\#f\#_{L^p(A)} := \left(\int_A |f|^p \right)^{\frac{1}{p}} = \left(\frac{1}{|A|} \int_A |f|^p \right)^{\frac{1}{p}}.$$

FIGURE 4. Proof Strategy when $d \geq 3$

Lemma 2.10 (Gerhing's Lemma, Proposition 1.3 of [20]). *Let $\theta \in [0, 1)$. Assume that*

$$\|v\|_{L^p(Q_{R/2}(z))} \leq \theta \|v\|_{L^p(Q_R(z))} + C_2 \|v\|_{L^1(Q_R(z))} + C_2 \|g\|_{L^p(Q_R(z))}$$

holds for any $z = (t, x) \in \mathbb{R}^{d+1}$ and $R > 0$, where $Q_r(z) = (t - R^2, t) \times B_r(x)$. Then, there exists $q = q(d, p, \theta, C_2) > p$ such that for any $z \in \mathbb{R}^{d+1}$ and $R > 0$,

$$\|v\|_{L^q(Q_{R/2}(z))} \leq C \|v\|_{L^p(Q_R(z))} + C \|g\|_{L^q(Q_R(z))}, \quad (2.23)$$

where C only depends on d, p, θ, C_2 and q .

Lemma 2.11. *Let $d = 2$. There is a constant C such that for any $R > 0$ and $u \in W^{1,2}(B_R)$,*

$$\|u\|_{L^4(B_R)}^4 \leq C \|u\|_{L^2(B_R)}^2 \int_{B_R} (R^{-2}|u|^2 + |\nabla u|^2). \quad (2.24)$$

Proof. Let E be the standard extension operator in the theory of Sobolev space (see [14]), which continuously maps $L^2(B_1)$ and $W^{1,2}(B_1)$ to $L^2(\mathbb{R}^2)$ and $W^{1,2}(\mathbb{R}^2)$, respectively. By the classical Ladyžhenskaya inequality on \mathbb{R}^2 , we have

$$\|u\|_{L^4(B_1)}^4 \leq \|Eu\|_{L^4(\mathbb{R}^2)}^4 \leq C \|\nabla Eu\|_{L^2(\mathbb{R}^2)}^2 \|Eu\|_{L^2(\mathbb{R}^2)}^2 \leq C \|u\|_{W^{1,2}(B_1)}^2 \|u\|_{L^2(B_1)}^2.$$

The desired inequality follows by the above estimate and scaling. \square

The following lemma is crucial for the proof of Theorem 2.7.

Lemma 2.12 (Reverse Hölder estimate). *Let $d = 2$, $\rho > 0$, and u be a weak solution to (2.13) in Q_ρ with $c = 0$. Then, there exists a constant C_3 only depending on $\|b\|_{L_t^\infty L_x^{2,\infty}(Q_\rho)}$ such that for any $R \in (0, \rho]$ and $\varepsilon \in (0, 1)$,*

$$\begin{aligned}
\iint_{Q_{\frac{R}{2}}} |\nabla u|^2 &\leq \varepsilon \left(1 + \|u\|_{L_t^\infty L_x^2(Q_\rho)}^2\right) \iint_{Q_R} |\nabla u|^2 \\
&\quad + C_3 \varepsilon^{-\frac{3}{2}} R^{-2} \left(\iint_{Q_R} |\nabla u|^{\frac{4}{3}}\right)^{\frac{3}{2}} + C_3 \varepsilon^{-\frac{1}{3}} \iint_{Q_R} |f|^{\frac{4}{3}},
\end{aligned}$$

for any $u \in L_t^\infty L_x^2(Q_\rho)$ and $f \in L_{t,x}^{\frac{4}{3}}(Q_R)$.

Proof. We use the cut-off function $\varphi \in C_c^\infty(B_1)$ satisfying $\varphi \in [0, 1]$ and $\varphi|_{B_{\frac{1}{2}}} = 1$, and let $\phi \in C_c^\infty((-1, 1))$ satisfy $\phi \in [0, 1]$ and $\phi|_{[-\frac{1}{2}, \frac{1}{2}]} = 1$. Let

$$\varphi_R(x) := \varphi\left(\frac{x}{R}\right), \quad \phi_R(t) := \phi\left(\frac{t}{R^2}\right),$$

and

$$\underline{u}_R := \left(\int_{B_R} u \varphi_R^{10} \right) \left(\int_{B_R} \varphi_R^{10} \right)^{-1}, \quad \bar{u}_R := u - \underline{u}_R.$$

For simplicity, the subscript R is omitted below. Multiplying both sides of (2.11) by $\bar{u}\varphi^{10}\phi^2$ and using the integration-by-parts formula, we have

$$\begin{aligned} \iint_{Q_R} \partial_t u \bar{u} \varphi^{10} \phi^2 &= \iint_{Q_R} \partial_t (u - \underline{u})(u - \underline{u}) \varphi^{10} \phi^2 \\ &= \frac{1}{2} \iint_{Q_R} \partial_t (\bar{u}^2 \varphi^{10} \phi^2) - \iint_{Q_R} \bar{u}^2 \varphi^{10} \phi \phi', \end{aligned}$$

and

$$- \iint_{Q_R} \Delta u \bar{u} \varphi^{10} \phi^2 = \iint_{Q_R} |\nabla u|^2 \varphi^{10} \phi^2 + 10 \iint_{Q_R} \bar{u} \varphi^9 \phi^2 \nabla u \cdot \nabla \varphi.$$

Since b is divergence free, we get

$$\iint_{Q_R} b \cdot \nabla u \bar{u} \varphi^{10} \phi^2 = \frac{1}{2} \iint_{Q_R} \varphi^{10} \phi^2 b \cdot \nabla (\bar{u}^2) = -5 \iint_{Q_R} \bar{u}^2 \varphi^9 \phi^2 b \cdot \nabla \varphi.$$

Thus,

$$\begin{aligned} & \underbrace{\sup_{t \in (-R^2, 0)} \int_{B_R} \bar{u}^2(t) \varphi^{10} \phi^2(t)}_{=: J_1} + \underbrace{\iint_{Q_R} |\nabla u|^2 \varphi^{10} \phi^2}_{=: J_2} \\ & \leq \frac{C}{R^2} \iint_{Q_R} \bar{u}^2 \varphi^{10} \phi + \frac{C}{R} \iint_{Q_R} |\nabla u| \bar{u} \varphi^9 \phi^2 + \frac{C}{R} \iint_{Q_R} |b| \bar{u}^2 \varphi^9 \phi^2 + C \iint_{Q_R} f \bar{u} \varphi^{10} \phi^2 \\ & \leq \frac{1}{2} \iint_{Q_R} |\nabla u|^2 \varphi^{10} \phi^2 + C \delta^{-\frac{1}{3}} \iint_{Q_R} |f|^{\frac{4}{3}} + \frac{\delta}{2} \iint_{Q_R} |\bar{u}|^4 \quad (\forall \delta > 0) \\ & \quad + \underbrace{\frac{C}{R^2} \iint_{Q_R} \bar{u}^2 \varphi^7 \phi}_{=: I_1} + \underbrace{\frac{C}{R} \iint_{Q_R} |b| \bar{u}^2 \varphi^9 \phi^2}_{=: I_2}. \end{aligned} \tag{2.25}$$

For I_1 , by Hölder's inequality and the Poincaré-type inequality (B.8), for any $\delta > 0$,

$$\begin{aligned} I_1 &= \frac{C}{R^2} \iint_{Q_R} \bar{u}^2 \varphi^7 \phi \leq \frac{C}{R^2} \iint_{Q_R} (\bar{u}^{\frac{2}{3}} \varphi^{\frac{10}{3}} \phi^{\frac{2}{3}}) \bar{u}^{\frac{4}{3}} \\ &\leq \frac{C}{R^2} \int_{-R^2}^0 \left(\int_{B_R} \bar{u}^2(t) \varphi^{10} \phi^2(t) \right)^{\frac{1}{3}} \left(\int_{B_R} \bar{u}^2(t) \right)^{\frac{2}{3}} dt \\ &\leq \delta \sup_{t \in (-R^2, 0)} \int_{B_R} \bar{u}^2(t) \varphi^{10} \phi^2(t) + \delta^{-\frac{1}{2}} \left[\int_{-R^2}^0 R^{-2} \left(\int_{B_R} \bar{u}^2(t) \right)^{\frac{2}{3}} dt \right]^{\frac{3}{2}} \\ &\leq \delta J_1 + C \delta^{-\frac{1}{2}} R^{-2} \left(\iint_{Q_R} |\nabla u|^{\frac{4}{3}} \right)^{\frac{3}{2}}. \end{aligned} \tag{2.26}$$

Regarding I_2 , by the Hölder inequality (A.2), the Ladyženskaja-type inequality for Lorentz spaces (B.6) and (2.26), we obtain that for any $\varepsilon > 0$,

$$\begin{aligned}
I_2 &\leq \frac{C}{R} \|b\|_{L_t^\infty L_x^{2,\infty}(Q_R)} \|\bar{u} \varphi^{\frac{9}{2}} \phi\|_{L_t^2 L_x^{4,2}(Q_R)}^2 \\
&\leq \frac{C}{R} \|b\|_{L_t^\infty L_x^{2,\infty}(Q_R)} \int_{-R^2}^0 \|\bar{u}(t) \varphi^{\frac{9}{2}} \phi(t)\|_{L_x^{4,2}(B_R)}^2 dt \\
&\leq \frac{C}{R} \|b\|_{L_t^\infty L_x^{2,\infty}(Q_R)} \int_{-R^2}^0 \|\bar{u}(t) \varphi^{\frac{9}{2}} \phi(t)\|_{L_x^2} \left\| \nabla \left(\bar{u}(t) \varphi^{\frac{9}{2}} \phi(t) \right) \right\|_{L_x^2} dt \\
&\leq \frac{C}{\varepsilon R^2} \left(1 + \|b\|_{L_t^\infty L_x^{2,\infty}(Q_R)}^2 \right) \iint_{Q_R} \bar{u}^2 \varphi^7 \phi + \frac{\varepsilon}{2} \iint_{Q_R} |\nabla u|^2 \\
&\leq C \varepsilon^{-1} \delta J_1 + C \varepsilon^{-1} \delta^{-\frac{1}{2}} R^{-2} \left(\iint_{Q_R} |\nabla u|^{\frac{4}{3}} \right)^{\frac{3}{2}} + \frac{\varepsilon}{2} \iint_{Q_R} |\nabla u|^2.
\end{aligned} \tag{2.27}$$

Combining (2.25)-(2.27), we get

$$\begin{aligned}
(1 - C_4 \delta / \varepsilon) J_1 + J_2 &\leq \varepsilon \iint_{Q_R} |\nabla u|^2 + \delta \iint_{Q_R} |\bar{u}|^4 \\
&\quad + C_4 \varepsilon^{-1} \delta^{-\frac{1}{2}} R^{-2} \left(\iint_{Q_R} |\nabla u|^{\frac{4}{3}} \right)^{\frac{3}{2}} + C_4 \delta^{-\frac{1}{3}} \iint_{Q_R} |f|^{\frac{4}{3}},
\end{aligned}$$

where C_4 only depends on $\|b\|_{L_t^\infty L_x^{2,\infty}(Q_\rho)}$.

Using (2.24) and the Poincaré-type inequality (B.7), we have

$$\begin{aligned}
\iint_{Q_R} |\bar{u}|^4 &\leq C \int_{-R^2}^0 dt \int_{B_R} |\bar{u}|^2(t) \left(\int_{B_R} R^{-2} |\bar{u}(t)|^2 + |\nabla \bar{u}(t)|^2 \right) \\
&\leq \int_{-R^2}^0 dt \int_{B_R} |\bar{u}|^2(t) \int_{B_R} |\nabla \bar{u}(t)|^2 \\
&\leq C \sup_{t \in (-R^2, 0)} \left(\int_{B_R} |\bar{u}|^2(t) \right) \iint_{Q_R} |\nabla u|^2 \\
&\leq C_5 \sup_{t \in (-R^2, 0)} \left(\int_{B_R} u^2(t) \right) \iint_{Q_R} |\nabla u|^2,
\end{aligned}$$

where C is a constant independent of δ and ε . Thus,

$$\begin{aligned}
(1 - C_4 \delta / \varepsilon) J_1 + J_2 &\leq \varepsilon \iint_{Q_R} |\nabla u|^2 + \delta C_5 \sup_{t \in (-R^2, 0)} \left(\int_{B_R} u^2(t) \right) \iint_{Q_R} |\nabla u|^2 \\
&\quad + C_4 \varepsilon^{-1} \delta^{-\frac{1}{2}} R^{-2} \left(\iint_{Q_R} |\nabla u|^{\frac{4}{3}} \right)^{\frac{3}{2}} + C_4 \delta^{-\frac{1}{3}} \iint_{Q_R} |f|^{\frac{4}{3}}.
\end{aligned} \tag{2.28}$$

For any $\varepsilon \in (0, 1)$, by choosing $\delta = \varepsilon / (C_4 + C_5)$, we obtain the desired estimate. \square

Lemma 2.13 (Meyers-type estimate). *Let $d = 2$. There exists a universal constant $\gamma > 2$ only depending on $\|b\|_{L_t^\infty L_x^{2,\infty}}$ such that for any weak solution to (2.11) on $[0, T] \times \mathbb{R}^d$ with $c = 0$, it holds*

$$\|\nabla u\|_{L_{t,x}^\gamma} \leq C \|f\|_{L_{t,x}^{\frac{4}{3}}}^{\frac{1}{3}} \|f\|_{L_{t,x}^{\frac{2\gamma}{3}}}^{\frac{2}{3}}, \tag{2.29}$$

for any $f \in L_{t,x}^{\frac{4}{3}} \cap L_{t,x}^{\frac{2\gamma}{3}}$. Here the constant C only depends on γ and $\|b\|_{L_t^\infty L_x^{2,\infty}}$.

Proof. As before, we assume that $b \in C_b^\infty(\mathbb{R}^d)$ with $\operatorname{div} b = 0$, $f \in C_c^\infty(\mathbb{R}^d)$. Then (2.11) admits a classical solution u that can be extended continuously to $(-\infty, T] \times \mathbb{R}^d$ by setting $u(t, \cdot) = 0$ for all $t \leq 0$.

Let

$$\tilde{f} = f/\|f\|_{L_{t,x}^{\frac{4}{3}}}, \quad \tilde{u} = u/\|f\|_{L_{t,x}^{\frac{4}{3}}}.$$

By the linearity of (2.11), \tilde{u} also satisfies (2.11) with f replaced by \tilde{f} . Thanks to Lemma 2.12 and Proposition 2.4, for any $R > 0$ and $z \in (-\infty, T) \times \mathbb{R}^2$, it holds that

$$\begin{aligned} \iint_{Q_{\frac{R}{2}}(z)} |\nabla \tilde{u}|^2 &\leq \varepsilon (1 + C_1) \iint_{Q_R(z)} |\nabla \tilde{u}|^2 \\ &+ C_3 \varepsilon^{-\frac{3}{2}} R^{-2} \left(\iint_{Q_R(z)} |\nabla \tilde{u}|^{\frac{4}{3}} \right)^{\frac{3}{2}} + C_3 \varepsilon^{-\frac{1}{3}} \iint_{Q_R(z)} |\tilde{f}|^{\frac{4}{3}}. \end{aligned} \quad (2.30)$$

Letting $\varepsilon := 2^{-10} (1 + C_1)^{-1}$ and

$$v := |\nabla \tilde{u}|^{\frac{4}{3}}, \quad g := |\tilde{f}|^{\frac{8}{9}}, \quad p := \frac{3}{2},$$

we obtain that

$$\|v\|_{L^p(Q_{R/2})} \leq \frac{1}{2} \|v\|_{L^p(Q_R)} + C_6 \|v\|_{L^1(Q_R)} + C_6 \|g\|_{L^p(Q_R)},$$

where C_6 only depends on $\|b\|_{L_t^\infty L_x^{2,\infty}}$. In the light of Lemma 2.10, there exists $\gamma > 2$ only depending on $\|b\|_{L_t^\infty L_x^{2,\infty}}$, such that for any $R > 0$,

$$\begin{aligned} \iint_{Q_{R/2}} |\nabla \tilde{u}|^\gamma &\leq C R^{4-2\gamma} \left(\iint_{Q_R} |\nabla \tilde{u}|^2 \right)^{\gamma/2} + C \iint_{Q_R} |\tilde{f}|^{\frac{2\gamma}{3}} \\ &\leq C R^{4-2\gamma} + C \iint_{Q_R} |\tilde{f}|^{\frac{2\gamma}{3}}, \end{aligned}$$

where C only depends on γ and $\|b\|_{L_t^\infty L_x^{2,\infty}}$. Letting $R \rightarrow \infty$, we obtain (2.29). \square

We are now ready to prove Theorem 2.7 in the case where $d = 2$.

Proof of Theorem 2.7. The proof relies on three steps to upgrade the integrability of the gradient of solutions. Again we focus on the smooth case where $b \in C_b^\infty(\mathbb{R}^d)$ with $\operatorname{div} b = 0$, $f \in C_c^\infty(\mathbb{R}^d)$, and additionally $c = 0$. Then (2.11) admits a classical solution u which can be extended continuously to $(-\infty, T] \times \mathbb{R}^d$ by setting $u(t, \cdot) = 0$ for all $t \leq 0$.

(i) Base estimates: Let \mathcal{T} be the solution map for the heat equation (2.20), i.e.,

$$\mathcal{T}(f)(t, x) := \int_0^t \int_{\mathbb{R}^2} h(t-s, y) f(s, y) \, dy \, ds,$$

where h is the heat kernel as in (1.2). Then

$$u = \mathcal{T}f + \mathcal{T}(b \cdot \nabla u) = \mathcal{T}f + \nabla \cdot \mathcal{T}(bu). \quad (2.31)$$

Set

$$\|f\| := \|f\|_{L_t^\infty L_x^1} + \|f\|_{L_t^\infty L_x^2}.$$

For any $\mu \in (1, 2)$ and $\nu \in (2, \infty)$, choosing

$$\eta = \left(\frac{1}{2\mu} + \frac{1}{2\nu} + \frac{1}{4} \right)^{-1} \in (1, 2),$$

and using the L^q - L^p estimate in Proposition 2.8 and the parabolic-type Sobolev embedding in Proposition 2.9, we have

$$\|\nabla\mathcal{T}(f)\|_{L_t^\nu L_x^\mu} \leq C \left(\|\partial_t \mathcal{T}(f)\|_{L_{t,x}^\eta} + \|\nabla^2 \mathcal{T}(f)\|_{L_{t,x}^\eta} \right) \leq C \|f\|_{L_{t,x}^\eta} \leq C \|f\|.$$

Moreover, using Theorem 2.1 and Lemma 2.5 we come to

$$\|u\|_{L_t^\infty L_x^{r,1}} \leq C \|f\| \quad \text{with} \quad r = \left(\frac{1}{\mu} - \frac{1}{2} \right)^{-1}.$$

Thus,

$$\|bu\|_{L_t^\nu L_x^\mu} \leq C \|b\|_{L_t^\infty L_x^{2,\infty}} \|u\|_{L_t^\infty L_x^{r,1}} \leq C \|f\|. \quad (2.32)$$

Therefore, using (2.31)-(2.32) and Proposition 2.8 again we obtain

$$\|\nabla u\|_{L_t^\nu L_x^\mu} \leq C \|\nabla^2 \mathcal{T}(bu)\|_{L_t^\nu L_x^\mu} + C \|\nabla \mathcal{T}(f)\|_{L_t^\nu L_x^\mu} \leq C \|f\|. \quad (2.33)$$

(ii) First upgradation: We note that the integrability exponent μ in the last estimate (2.33) is strictly less than two, which is insufficient to derive (2.19). The key idea in this step is to make use of the Meyers-type estimate to upgrade the integrability of the gradient of solutions.

More precisely, let $\gamma > 2$ be the integrability exponent as in Lemma 2.13 and $\varepsilon \in (0, \gamma^{-1})$ be a small parameter to be determined later. Set

$$\theta = 1 - \varepsilon\gamma \in (0, 1), \quad \mu = \frac{1 - \varepsilon\gamma}{1/2 - (\gamma/4 + 1/2)\varepsilon} \in (1, 2), \quad \nu > \frac{1 - \varepsilon\gamma}{(\gamma/4 - 1/2)\varepsilon} \vee 1.$$

Combining (2.33) and the Meyers-type estimate (2.29) together, we can upgrade the integrability of ∇u by

$$\|\nabla u\|_{L_t^q L_x^{l,1}} \leq \|\nabla u\|_{L_t^\nu L_x^\mu}^\theta \|\nabla u\|_{L_{t,x}^\gamma}^{1-\theta} \leq C \|f\|,$$

where

$$\frac{1}{l} = \frac{\theta}{\mu} + \frac{1-\theta}{\gamma}, \quad \frac{1}{q} = \frac{\theta}{\nu} + \frac{1-\theta}{\gamma} > \varepsilon.$$

Note that, by our choice of parameters, one has

$$\frac{1}{l} = \frac{1}{2} - \frac{\gamma\varepsilon}{4} + \frac{\varepsilon}{2} < \frac{1}{2}, \quad \varepsilon < \frac{1}{q} < \frac{\gamma\varepsilon}{4} + \frac{\varepsilon}{2}. \quad (2.34)$$

In particular, the integrability exponent l of the energy is now raised above $2 = d$. Put

$$p = \frac{2l}{2+l}.$$

Then

$$\|b \cdot \nabla u\|_{L_t^q L_x^p} \leq C \|b\|_{L_t^\infty L_x^{2,\infty}} \|\nabla u\|_{L_t^q L_x^{l,1}} \leq C \|f\|. \quad (2.35)$$

Note that

$$1 + \varepsilon > \frac{1}{q} + \frac{1}{p} = \frac{1}{2} + \frac{1}{l} + \frac{1}{q} > 1 + \varepsilon \left(\frac{3}{2} - \frac{\gamma}{4} \right) > 1 \quad (\text{if } \gamma < 6), \quad (2.36)$$

due to (2.34).

(iii) Second upgradation: We still need to further upgrade the integrability of ∇u . For this purpose, using the L^q - L^p estimate in Proposition 2.8 again and (2.35), we see that

$$\|\nabla^2 u\|_{L_t^q L_x^p} \leq C \left(\|b \cdot \nabla u\|_{L_t^q L_x^p} + \|f\|_{L_t^q L_x^p} \right) \leq C \|f\|.$$

By virtue of the fractional Gagliardo-Nirenberg estimate in Proposition B.2, we obtain the improved estimate

$$\|\nabla u\|_{L_t^{q_i} L_x^{l_i}} \leq \|\nabla^2 u\|_{L_t^q L_x^p}^{\delta_i} \|u\|_{L_t^\infty C^{\alpha_i}}^{1-\delta_i} \leq C \|f\|,$$

where $0 < \alpha_1 = \frac{1}{3}\alpha, \alpha_2 = \alpha, \delta_i = \frac{1-\alpha_i}{2-\alpha_i}$ and α is the constant in Proposition 2.6, and

$$\frac{1}{l_i} = \frac{\delta_i}{p}, \quad \frac{1}{q_i} = \frac{\delta_i}{q}.$$

Using the interpolation estimate (B.1) again we have

$$\|\nabla u\|_{L_t^q L_x^{l_1}} \leq C \left(\|\nabla u\|_{L_t^{q_1} L_x^{l_1}} + \|\nabla u\|_{L_t^{q_2} L_x^{l_2}} \right) \leq C \|f\|,$$

where

$$l := \frac{p}{\delta}, \quad q := \frac{q}{\delta} \quad \text{with} \quad 0 < \delta = \frac{1-\alpha/2}{2-\alpha/2} < \frac{1}{2}.$$

This further yields the improved integrability estimate

$$\|b \cdot \nabla u\|_{L_t^q L_x^p} \leq C \|b\|_{L_t^\infty L_x^{2,\infty}} \|\nabla u\|_{L_t^q L_x^{l_1}} \leq C \|f\|,$$

where

$$\frac{1}{p} := \frac{1}{l} + \frac{1}{2} = \frac{\delta}{p} + \frac{1}{2}.$$

Now choosing $\varepsilon < \frac{1}{\gamma} \wedge \left(\frac{1}{2\delta} - 1\right)$ and using (2.36) we infer that the improved integrability exponent (p, q) exactly lies in \mathcal{I} , that is,

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \delta \left(\frac{1}{2} + \frac{1}{l} + \frac{1}{q} \right) < \frac{1}{2} + \delta(1 + \varepsilon) < 1.$$

Thus, applying the L^q - L^p estimate in Proposition 2.8 again we obtain the desired estimate (2.19) in the case where $c = 0$.

Now, the case where $c \in L_t^\infty(L_x^1 \cap L_x^\infty)$ can be estimated easily by applying the second order Sobolev estimate (2.19) for $c = 0$ and (2.16):

$$\|u\|_{L_t^q W_x^{2,p}} + \|\partial_t u\|_{L_t^q L_x^p} \leq C \|cu + f\| \leq C \|u\|_{L_{t,x}^\infty} \|c\|_{L_t^\infty(L_x^1 \cap L_x^\infty)} + C \|f\| \leq C \|f\|,$$

where C only depends on $d, p, q, T, \|b\|_{L_t^\infty L_x^{d,\infty}}$ and $\|c\|_{L_t^\infty(L_x^1 \cap L_x^\infty)}$. The proof is complete. \square

3. SDES WITH ENDPOINT CRITICAL DRIFTS

In this section, we aim to establish Theorem 1.2 with the endpoint critical drifts. Without loss of generality, let us assume $s = 0$ in (SDE).

3.1. Existence. To begin with, let us start with the generalized Itô formula.

Lemma 3.1 (Generalized Itô's formula). *Let $(p, q) \in \mathcal{I}$. Suppose that X_t is a solution to (SDE) with $s = 0$ satisfying the following Krylov-type estimate*

$$\mathbf{E} \int_0^T f(X_t) dt \leq C \|f\|_{L_t^q L_x^p},$$

and $u \in L_t^q W_x^{2,p}$ with $\partial_t u \in L_t^q L_x^p$. Then, for any $0 \leq t' < t \leq T$, we have

$$u(t, X_t) - u(t', X_{t'}) = \int_{t'}^t (\partial_r + A_r) u(r, X_r) dr + \sqrt{2} \int_{t'}^t \nabla u(r, X_r) \cdot dW_r. \quad (3.1)$$

Proof. Take any mollifier $\varrho \in C_c^\infty(\mathbb{R}^{d+1})$ such that $\int \varrho = 1$, and let $\varrho^n(t, x) = n^{d+2} \varrho(n^2 t, nx)$. Let $u^n = u * \varrho^n \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$. Itô's formula gives that

$$u^n(t, X_t) - u^n(t', X_{t'}) = \int_{t'}^t (\partial_r + A_r) u^n(r, X_r) dr + \sqrt{2} \int_{t'}^t \nabla u^n(r, X_r) \cdot dW_r, \quad (3.2)$$

where A_r is the Kolmogorov operator as in Section 2. Since

$$\begin{aligned} \|(\partial_t + A_t)(u - u^n)\|_{L_t^q L_x^p} &\leq \|\partial_t u - \partial_t u * \varrho^n\|_{L_t^q L_x^p} + \|\Delta u - \Delta u * \varrho^n\|_{L_t^q L_x^p} \\ &\quad + \|b\|_{L_t^\infty L_x^{d,\infty}} \|\nabla u - \nabla u * \varrho^n\|_{L_t^q L_x^{\frac{dp}{d-p}, p}} \\ &\leq C \|\partial_t u - \partial_t u * \varrho^n\|_{L_t^q L_x^p} + C \|\nabla^2 u - \nabla^2 u * \varrho^n\|_{L_t^q L_x^p} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, by the Krylov estimate one has

$$\int_{t'}^t (\partial_r + A_r) u^n(r, X_r) dr \rightarrow \int_{t'}^t (\partial_r + A_r) u(r, X_r) dr \quad \text{in } L^1(\Omega, \mathbf{P}).$$

Moreover, noting that

$$\frac{d}{p} + \frac{2}{q} < \frac{d}{2p} + \frac{1}{q} + 1,$$

by the parabolic-type Sobolev embedding in Proposition 2.9, there exists $l > 2q$ such that

$$\|\nabla u - \nabla u^n\|_{L_t^l L_x^{2p}} \leq C \left(\|\partial_t u - \partial_t u^n\|_{L_t^q L_x^p} + \|\nabla^2 u - \nabla^2 u^n\|_{L_t^q L_x^p} \right) \rightarrow 0.$$

This along with Doob's inequality and Krylov's estimate yields that

$$\begin{aligned} &\mathbf{E} \sup_{0 \leq t' < t \leq T} \left| \int_{t'}^t \nabla u^n(r, X_r) \cdot dW_r - \int_{t'}^t \nabla u(r, X_r) \cdot dW_r \right|^2 \\ &\leq C \mathbf{E} \int_0^T |\nabla u^n(r, X_r) - \nabla u(r, X_r)|^2 dr \\ &\leq C \|(\nabla u^n - \nabla u)\|_{L_t^{2q} L_x^{2p}}^2 \leq C \|(\nabla u^n - \nabla u)\|_{L_t^l L_x^{2p}}^2 \rightarrow 0, \end{aligned}$$

where the last step is due to $l > 2q$. Therefore, the right-hand side of (3.2) converges to that of (3.1). Taking into account that for any $0 < \alpha < (2 - \frac{d}{p} - \frac{2}{q}) \wedge 1$,

$$\|u\|_{C^\alpha} \leq C \left(\|\partial_t u\|_{L_t^q L_x^p} + \|\nabla^2 u\|_{L_t^q L_x^p} \right)$$

(see [35]), we also infer that the left-side of (3.2) converges to that of (3.1), and thus finish the proof. \square

We are now in a position to provide the

Proof of the existence part of Theorem 1.2: Let $s = 0$ for simplicity. Take any mollifier $\varrho \in C_c^\infty(\mathbb{R}^{d+1})$ such that $\int \varrho = 1$, and let $\varrho^n(t, x) = n^{d+2} \varrho(n^2 t, nx)$. Set

$$b_1 = b \mathbf{1}_{\{|b|>1\}}, \quad b_2 = b \mathbf{1}_{\{|b|\leq 1\}} \quad \text{and} \quad b_i^n = b_i * \varrho^n, \quad b^n = b * \varrho^n.$$

Let

$$p_1 = 3d/4 \quad \text{and} \quad p_2 = 2d$$

and $(p_i, q_i) \in \mathcal{I}$, $i = 1, 2$. Noting that

$$\|b_1\|_{L_t^\infty L_x^{p_1}}^{p_1} \leq C \|b\|_{L_t^\infty L_x^{d,\infty}}^d \int_1^\infty \lambda^{p_1-1-d} d\lambda \leq C \|b\|_{L_t^\infty L_x^{d,\infty}}^d$$

and

$$\|b_2\|_{L_t^\infty L_x^{p_2}}^{p_2} \leq C \|b_2\|_{L_t^\infty L_x^{d,\infty}}^d \|b_2\|_{L_{t,x}^\infty}^{p_2-d} \leq C \|b\|_{L_t^\infty L_x^{d,\infty}}^d,$$

we have that $C_b^\infty \ni b_i^n \rightarrow b_i$ in $L_t^{q_i} L_x^{p_i}$. Since $b^n \in C_b^\infty$, the classical SDE theory guarantees the existence of a unique strong solution X^n to (SDE), with b replaced by b^n , starting from $s = 0$. Moreover, by the classical parabolic PDE theory, for any $(p, q) \in \mathcal{I}$, $t_1 \in (0, T]$ and $f \in L_t^q L_x^p$, there exists a unique solution v^n in $L_t^q W_x^{2,p}$ satisfying

$$\partial_t v^n + \Delta v^n + b^n \cdot \nabla v^n + f = 0, \quad v^n(t_1) = 0.$$

Then, by Itô's formula and Proposition 2.6, there exist $\alpha \in (0, 1/2)$, $C > 0$ such that for any $0 \leq t_0 < t_1 \leq T$,

$$\sup_n \mathbf{E} \int_{t_0}^{t_1} f(t, X_t^n) dt \leq \sup_{x \in \mathbb{R}^d} |v^n(t_0, x)| \leq C (t_1 - t_0)^\alpha \|f\|_{L_t^q L_x^p}. \quad (3.3)$$

Now let $\tau \in [0, T]$ be any bounded stopping time. Since

$$X_{(\tau+\delta)\wedge T}^n - X_\tau^n = \int_\tau^{(\tau+\delta)\wedge T} b_n(t, X_t^n) dt + \sqrt{2}(W_{(\tau+\delta)\wedge T} - W_\tau), \quad \delta > 0,$$

applying (3.3) to $f = b_1^n$ we get

$$\mathbf{E} \int_\tau^{(\tau+\delta)\wedge T} |b_1^n|(t, X_t^n) dt \leq C \delta^\alpha \left(\|b_1^n\|_{L_t^\infty L_x^{p_1}} + \|b_2^n\|_{L_{t,x}^\infty} \right) \leq C \delta^\alpha.$$

So one derives that

$$\mathbf{E} \sup_{0 \leq u \leq \delta} |X_{\tau+u}^n - X_\tau^n| \leq \mathbf{E} \int_\tau^{\tau+\delta} |b_n|(t, X_t^n) dt + \sqrt{2} \mathbf{E} \sup_{0 \leq u \leq \delta} |W_{\tau+u} - W_\tau| \leq C \delta^\alpha,$$

where α and $C > 0$ are independent of n . Hence, by [53, Lemma 2.7], we obtain

$$\sup_n \mathbf{E} \left(\sup_{t \in [0, T]; u \in [0, \delta]} |X_{(t+u)\wedge T}^n - X_t^n|^{1/2} \right) \leq C \delta^\alpha,$$

which along with Chebyshev's inequality yields that for any $\varepsilon > 0$,

$$\lim_{\delta \rightarrow 0} \sup_n \mathbf{P} \left(\sup_{t \in [0, T]; u \in [0, \delta]} |X_{(t+u)\wedge T}^n(x) - X_t^n(x)| > \varepsilon \right) = 0.$$

This implies the tightness of the probability laws $\{\mathbf{P} \circ (X^n, W)^{-1}\}$ in $\mathcal{P}(C^{\otimes 2}([0, T]; \mathbb{R}^d))$.

Hence, by Skorokhold's representation theorem, there exist a probability space $(\Omega, \mathcal{F}, \mathbf{Q})$, and $C^{\otimes 2}([0, T]; \mathbb{R}^d)$ -valued random variables (Y^n, B^n) and (Y, B) on $(\Omega, \mathcal{F}, \mathbf{Q})$ such that

$$(X^n, W) \stackrel{d}{=} (Y^n, B^n)$$

and up to a subsequence $(Y^n, B^n) \rightarrow (Y, B)$, \mathbf{Q} -a.s.. it is clear that B is a Brownian motion. Moreover, for any $(p, q) \in \mathcal{I}$ and $f \in L_t^q L_x^p$,

$$\mathbf{E}_{\mathbf{Q}} \int_0^T f(t, Y_t^n) dt, \mathbf{E}_{\mathbf{Q}} \int_0^T f(t, Y_t) dt \leq C \|f\|_{L_t^q L_x^p}, \quad (3.4)$$

where C is independent of n .

In order to prove that (Y, B) is a weak solution to (SDE), we only need to show that

$$\int_0^T |b(t, Y_t) - b^n(t, Y_t^n)| dt \rightarrow 0, \quad \text{in } \mathbf{Q} \text{ - probability.}$$

To this end, using (3.4) we derive that for any fixed $N, n \geq 1$,

$$\begin{aligned} & \mathbf{E}_{\mathbf{Q}} \int_0^T |b(t, Y_t) - b^n(t, Y_t^n)| dt \\ & \leq \sum_{i=1}^2 \mathbf{E}_{\mathbf{Q}} \int_0^T |b_i(t, Y_t) - b_i^N(t, Y_t)| dt + \sum_{i=1}^2 \mathbf{E}_{\mathbf{Q}} \int_0^T |b_i^N(t, Y_t^n) - b_i^n(t, Y_t^n)| dt \\ & \quad + \mathbf{E}_{\mathbf{Q}} \int_0^T |b^N(t, Y_t) - b^n(t, Y_t^n)| dt \\ & \leq C \sum_{i=1}^2 \|b_i - b_i^N\|_{L_t^{q_i} L_x^{p_i}} + C \sum_{i=1}^2 \|b_i^N - b_i^n\|_{L_t^{q_i} L_x^{p_i}} + \int_0^T \mathbf{E}_{\mathbf{Q}} |b^N(t, Y_t^n) - b^n(t, Y_t)| dt. \end{aligned}$$

Letting $n \rightarrow \infty$ and then $N \rightarrow \infty$, we thus obtain the desired existence result.

At last, by Theorem 2.1, for each n , X^n admits a transition density function $p_{s,t}^n(x, y)$ that satisfies (AE) with the universal constant C . Therefore, each limit point of $\{X^n\}$ also satisfies estimates (AE) and (KE). \square

3.2. Uniqueness. Let us continue to prove the uniqueness part of Theorem 1.2.

Proof of the uniqueness part of Theorem 1.2: Suppose that $(\Omega^1, \mathcal{F}^1, \mathcal{F}_t^1, \mathbf{P}^1; X^1, W^1)$ and $(\Omega^2, \mathcal{F}^2, \mathcal{F}_t^2, \mathbf{P}^2; X^2, W^2)$ are two weak solutions to (SDE).

Let us first consider the case where each X^i ($i = 1, 2$) satisfies the Krylov-type estimate (KE), with $(p, q) = (\mathbf{p}, \mathbf{q}) \in \mathcal{I}$. Here (\mathbf{p}, \mathbf{q}) is the same pair as in Theorem 2.7.

For any $c, f \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$, in view of Theorem 2.7, equation (2.12) has a solution v satisfying that $v \in L_t^q W_x^{2,\mathbf{p}}$ and $\partial_t v \in L_t^q L_x^{\mathbf{p}}$ with $d/\mathbf{p} + 2/\mathbf{q} < 2$. Then, by the generalized Itô's formula (3.1),

$$\begin{aligned} & d \left(\exp \left(\int_0^t c(s, X_s^i) ds \right) v(t, X_t^i) \right) \\ & = \exp \left(\int_0^t c(s, X_s^i) ds \right) f(t, X_t^i) dt + \sqrt{2} \exp \left(\int_0^t c(s, X_s^i) ds \right) \nabla v(t, X_t^i) \cdot dW_t^i. \end{aligned}$$

Taking expectation one gets

$$v(0, x) = \mathbf{E}_{\mathbf{P}^i} \int_0^T \exp \left(\int_0^t c(s, X_s^i) ds \right) f(t, X_t^i) dt.$$

Thus, by the density argument, this yields that for any $\lambda > 0$,

$$\mathbf{E}_{\mathbf{P}^1} \int_0^T \exp \left(\int_0^t c(s, X_s^1) ds \right) e^{-\lambda t} dt = \mathbf{E}_{\mathbf{P}^2} \int_0^T \exp \left(\int_0^t c(s, X_s^2) ds \right) e^{-\lambda t} dt,$$

and so

$$\mathbf{P}^1 \circ \left(\exp \left(\int_0^t c(s, X_s^1) ds \right) \right)^{-1} = \mathbf{P}^2 \circ \left(\exp \left(\int_0^t c(s, X_s^2) ds \right) \right)^{-1}, \quad t \in [0, T].$$

Therefore, again by the density argument, for every $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \in [0, T]$, the moment generating functions of $\int_0^t \mathbf{1}_A(X_s^i) ds$ are identical for $i = 1, 2$, which implies the uniqueness in law $\mathbf{P}^1 \circ (X^1)^{-1} = \mathbf{P}^2 \circ (X^2)^{-1}$. This together with the existence part of Theorem 1.2 imply that the unique solution admits a transition density function satisfying (AE).

Now, suppose that X is a weak solution to (SDE) satisfying the generalized Krylov estimate (KE') with $s = 0$, where the pair (\mathbf{p}, \mathbf{q}) is the same as in Theorem 2.7. We claim that it indeed satisfies (KE) for $(p, q) = (\mathbf{p}, \mathbf{q})$ and $s = 0$. Thus, by the arguments in the previous case, we obtain the desired assertion.

To this end, note that for any $\delta \in (0, T)$, the process $(X_t)_{t \in [\delta, T]}$ satisfies

$$X_t = X_\delta + \int_\delta^t b(s, X_s) ds + \sqrt{2}(W_t - W_\delta),$$

and (KE) with $(p, q) = (\mathbf{p}, \mathbf{q}) \in \mathcal{I}$, $C = C_\delta$ and s replaced by δ . Then, as in the previous case, one has

$$\mathbf{E} \int_\delta^T f(t, X_t) dt \leq \mathbf{E} v(\delta, X_\delta) \leq \|v\|_{L_{t,x}^\infty} \leq C \|f\|_{L_t^q L_x^p}.$$

Here $v \in L_t^q L_x^p$ is the solution to (2.12) with $c = 0$. Note that, thanks to Theorem 2.7, the constant C on the right-hand side of the inequality above is universal, i.e., independent of δ . Thus, taking the limit as $\delta \rightarrow 0$, we conclude that X satisfies the Krylov estimate (KE) with $(p, q) = (\mathbf{p}, \mathbf{q})$ and $s = 0$. Therefore, the proof for the uniqueness part of Theorem 1.2 is complete.

The proof for the Markov property of $(\mathbb{P}_{s,x}, X)$ is standard (cf. [50]), so we omit this here. \square

In the end of this section, we also have the well-posedness of the following linear Fokker-Planck equation for all dimensions $d \geq 2$,

$$\partial_t \rho = \Delta \rho - \operatorname{div}(b\rho), \quad \rho|_{t=0} = \zeta \in \mathcal{M}(\mathbb{R}^d). \quad (3.5)$$

Definition 3.1. *We say that $\rho \in C_t \mathcal{M}_x$ is a distributional solution to the Fokker-Planck equation (3.5), if $b \in L^1(\rho)$ and for any $\phi \in C_c^\infty([0, T] \times \mathbb{R}^d)$, it holds that*

$$\langle \rho(t), \phi(t) \rangle - \langle \zeta, \phi(0) \rangle = \int_0^t \langle \rho(s), \partial_s \phi(s) + \Delta \phi(s) + b(s) \cdot \nabla \phi(s) \rangle ds, \quad t \in [0, T], \quad (3.6)$$

where $\langle \cdot, \cdot \rangle$ denotes the integration over \mathbb{R}^d .

Proposition 3.2 (Well-posedness of linear FPE). *Let $d \geq 2$. Assume that the drift b satisfies (Ab). Then, the Fokker-Planck equation (3.5) has a unique distributional solution $\rho \in C_t \mathcal{M}_x$ satisfying*

$$\rho \in L^{q'}(\delta, T; L_x^{p'}), \quad \forall \delta \in (0, T),$$

where (\mathbf{p}, \mathbf{q}) is as in Theorem 1.2, and \mathbf{p}', \mathbf{q}' are the conjugate numbers of \mathbf{p} and \mathbf{q} , respectively.

Proof. By Theorem 1.2 and the linearity of (3.5), the function $\rho(t) := \int_{\mathbb{R}^d} p_{0,t}(x, y) \zeta(dx)$ is a distributional solution of (3.5), where $p_{0,t}$ is the transition density in Theorem 1.2.

Regarding the uniqueness, as in the proof for the uniqueness part of Theorem 1.2, we can assume $\rho \in C_t \mathcal{M}_x \cap L_t^{q'} L_x^{p'}$. Let $v \in L_t^q W_x^{2,p}$ with $\partial_t v \in L_t^q L_x^p$ be the solution to (2.12) with

$c = 0$ and $f \in C_c^\infty((0, T) \times \mathbb{R}^d)$. Then, using a standard limiting argument, it can be shown that (3.6) holds for $\rho \in C_t \mathcal{M}_x \cap L_t^{q'} L_x^p$ and ϕ replaced by v , which leads to

$$-\langle \zeta, v(0) \rangle = \int_0^T \langle \rho(s), f(s) \rangle ds, \quad \forall f \in C_c^\infty((0, T) \times \mathbb{R}^d),$$

thereby yielding the uniqueness of ρ . \square

4. CONSTRUCTION OF NON-UNIQUE SOLUTIONS

This section is devoted to the non-uniqueness problem of SDEs and Fokker-Planck equations in the supercritical regime.

4.1. SDEs with supercritical drifts. Let us first prove the non-uniqueness result in Theorem 1.3 by constructing a divergence free drift $b \in L^{p, \infty}(\mathbb{R}^d)$ with $d/2 < p < d$ and $d \geq 3$, such that (SDE) have at least two distinct weak solutions starting from the origin.

Proof of Theorem 1.3. The proof mainly proceeds in five steps.

Step 1: Construction of the drift. Let us first give the specific construction of the drift b in the Lorentz space $L_x^{p, \infty}$ with $p \in (d/2, d)$.

Let g be a non-negative smooth function on $[0, \infty)$ such that $g'(r) \geq 0$ for all $r \geq 0$, $g(r) = 0$ if $0 \leq r \leq 1/2$ and $g(r) = 1$ if $r > 1$. Let

$$\alpha := \frac{d}{p} \in (1, 2). \quad (4.1)$$

For any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ with $x_d > 0$, set

$$r := (x_1^2 + \dots + x_{d-1}^2)^{1/2} \quad \text{and} \quad H(x) := r^{d-1} x_d^{-\alpha} g(x_d/r).$$

We define the drift b by

$$\begin{aligned} b_d(x) &:= N r^{2-d} \partial_r H(x) \\ &= N(d-1) x_d^{-\alpha} g(x_d/r) - N r^{-1} x_d^{-\alpha+1} g'(x_d/r) \quad \text{if } x_d > 0 \end{aligned} \quad (4.2)$$

and for $1 \leq i \leq d-1$,

$$\begin{aligned} b_i(x) &:= -N x_i r^{1-d} \partial_{x_d} H(x) \\ &= N \alpha (x_i x_d^{-\alpha-1}) g(x_d/r) - N r^{-1} x_i x_d^{-\alpha} g'(x_d/r) \quad \text{if } x_d > 0. \end{aligned} \quad (4.3)$$

Set

$$b(x_1, \dots, x_{d-1}, x_d) := -b(x_1, \dots, x_{d-1}, -x_d) \quad \text{if } x_d < 0$$

and, if $x_d = 0$,

$$b(x_1, \dots, x_{d-1}, 0) := 0.$$

Next, we shall verify that the constructed drift b belongs to the Lorentz space $L_x^{p, \infty}$ and is divergence free $\operatorname{div} b = 0$.

To this end, we infer from (4.2)-(4.3) and the anti-symmetry of b about the x_d -axis that

$$|b(x)| \leq C |x_d|^{-\alpha} \mathbf{1}_{\{r < 2|x_d|\}}. \quad (4.4)$$

Thus, by the choice (4.1),

$$\|b\|_{L^{p, \infty}}^p = \sup_{t>0} t^p |\{x : |b(x)| > t\}| \leq C \sup_{t>0} t^p \left| B_{t^{-\frac{1}{\alpha}}} \right| \leq C < \infty.$$

Note that, b is smooth on $\mathbb{R}^d \setminus \{0\}$.

Moreover, by the construction, we have

$$\partial_d b_d(x) = Nr^{2-d} \partial_{rx_d}^2 H(x), \quad \text{for all } x_d > 0$$

and

$$\partial_i b_i(x) = -N[r^{1-d} + (1-d)x_i^2 r^{-d-1}] \partial_{x_d} H(x) + Nx_i^2 r^{-d} \partial_{rx_d}^2 H(x),$$

for all $x_d > 0$ and $1 \leq i \leq d-1$. Taking into account (4.4) and the anti-symmetry of b about the x_d -axis, we obtain that $\operatorname{div} b(x) = 0$ if $x \neq 0$.

In order to verify that $\operatorname{div} b = 0$ in the sense of distribution, we need to show that $\int b \cdot \nabla \varphi = 0$ for any $\varphi \in C_c^\infty(\mathbb{R}^d)$. Actually, by the integration-by-parts formula and (4.4), for any $\rho > 0$,

$$\begin{aligned} \int b \cdot \nabla \varphi &= \int_{B_\rho} b \cdot \nabla \varphi + \int_{\partial B_\rho} b \varphi \cdot d\vec{\sigma} - \int_{B_\rho^c} \operatorname{div} b \cdot \nabla \varphi \\ &= \int_{B_\rho} b \cdot \nabla \varphi + \int_{\partial B_\rho} b \varphi \cdot d\vec{\sigma} \leq C\rho^{-\alpha+d-1}, \end{aligned}$$

which tends to zero as $\rho \rightarrow 0^+$, due to the fact that $\alpha \in (1, 2)$ and $d \geq 3$.

Step 2: Contradiction arguments. Let $\Omega = C(\mathbb{R}_+; \mathbb{R}^d)$ and ω_t be the canonical process. We consider a continuous functional \mathcal{I} on $\mathcal{P}(\Omega)$ defined by

$$\mathcal{I}(\mathbb{P}) := \mathbb{E}_{\mathbb{P}} \left[\int_0^\infty e^{-t} f(\omega_t) dt \right],$$

where

$$f(x) := \operatorname{sgn}(x_d) g(|x_d|).$$

Since $g|_{[0, \frac{1}{2}]} \equiv 0$, f is a continuous function on \mathbb{R}^d . Thus, the map $\omega \mapsto \int_0^\infty e^{-t} f(\omega_t) dt$ is continuous from Ω to \mathbb{R} . This yields that \mathcal{I} is continuous on $\mathcal{P}(\Omega)$, i.e.,

$$\mathcal{I}(\mathbb{P}_n) \rightarrow \mathcal{I}(\mathbb{P}), \quad \text{if } \mathbb{P}_n \Rightarrow \mathbb{P}. \quad (4.5)$$

Thanks to [54, Theorem 1.1], there exists at least one weak solution to (SDE) satisfying (KE), for any $x \in \mathbb{R}^d$ and $s = 0$.

Below we shall prove the non-uniqueness of solutions by contradiction arguments. Assume that the law of weak solutions starting from the origin is unique in the Krylov class (KE). Noting that b is smooth and uniformly bounded on $B_\varepsilon^c(0)$ for any $\varepsilon > 0$, our assumption and [50, Theorem 6.1.2] imply that martingale solutions to (SDE) starting from any $x \in \mathbb{R}^d$ is unique in the Krylov class (KE).

Let \mathbb{P}_x denote the unique martingale solution starting from x . Our strategy is to find two sequences (x^n) and (y^n) in \mathbb{R}^d converging to zero such that

$$\mathbb{P}_{x^n} \Rightarrow \mathbb{P}_0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathcal{I}(\mathbb{P}_{x^n}) \geq p_0 > 0 \quad (4.6)$$

for some $p_0 > 0$, and

$$\mathbb{P}_{y^n} \Rightarrow \mathbb{P}_0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathcal{I}(\mathbb{P}_{y^n}) = 0. \quad (4.7)$$

Then, the continuity of \mathcal{I} and (4.6)-(4.7) yield that

$$0 < p_0 \leq \lim_{n \rightarrow \infty} \mathcal{I}(\mathbb{P}_{x^n}) = \mathcal{I}(\mathbb{P}_0) = \lim_{n \rightarrow \infty} \mathcal{I}(\mathbb{P}_{y^n}) = 0,$$

which leads to a contradiction.

In order to construct such sequences, we first note that, since both b and f are anti-symmetric about the hyperplane $\Pi = \{x \in \mathbb{R}^d : x_d = 0\}$,

$$\mathcal{I}(\mathbb{P}_y) = \mathbb{E}_y \left[\int_0^\infty e^{-t} f(\omega_t) dt \right] = 0, \quad \text{if } y_d = 0. \quad (4.8)$$

(Here we write \mathbb{E}_y for $\mathbb{E}_{\mathbb{P}_y}$ for simplicity.)

Let $\mathbb{R}^d \setminus \{0\} \ni z^n \rightarrow 0$. As shown in the proof of Theorem 1.2 (using [54, Theorem 2.2] instead of Proposition 2.6), $\{\mathbb{P}_{z^n}\}$ is tight in $\mathcal{P}(\Omega)$ and its accumulation points are martingale solutions to (SDE) with $s = 0$ and $x = 0$ and satisfy (KE). But by the uniqueness assumption, one has $\mathbb{P}_{z^n} \Rightarrow \mathbb{P}_0$, $n \rightarrow \infty$. Therefore, if $y^n \rightarrow 0$ with $(y^n)_d = 0$, then (4.8) implies (4.7).

It remains to construct the sequence $x^n \rightarrow 0$ verifying (4.6).

Step 3: Reduction to exit probability estimate. In order to find such $\{x^n\}$ satisfying (4.6), we define the cone

$$\mathcal{C}_{k,h} = \left\{ z \in \mathbb{R}^d : k(z_1^2 + \cdots + z_{d-1}^2)^{\frac{1}{2}} < z_d < h, k, h > 0 \right\}, \quad \mathcal{C}_k = \mathcal{C}_{k,\infty}.$$

Let

$$\bar{\tau} = \inf\{t > 0 : \omega_t \notin \mathcal{C}_{1,2}\} \quad \text{and} \quad \tau = \inf\{t > 0 : \omega_t \notin \mathcal{C}_1\}$$

be the first exit times of the canonical process from $\mathcal{C}_{1,2}$ and \mathcal{C}_1 , respectively. Also let

$$\sigma_0 = \inf\{t > 0 : (\omega_t)_d = 0\}, \quad \sigma_1 = \inf\{t > \bar{\tau} : (\omega_t)_d = 1\}$$

be the first hitting times of ω to the hyperplanes $\{x_d = 0\}$ and $\{x_d = 1\}$ after time $\bar{\tau}$, respectively. Let

$$\kappa \in (1, (d-1)/\alpha).$$

Claim: We have the exit probability estimate: there exists a positive constant p_1 such that

$$\inf_{x \in \mathcal{C}_{\kappa,1}} \mathbb{P}_x(\bar{\tau} < 1 \wedge \tau, \sigma_1 > 1 + \bar{\tau}) \geq p_1 > 0. \quad (4.9)$$

Intuitively, estimate (4.9) shows that the solution trajectories starting from $x \in \mathcal{C}_{\kappa,1}$ are likely to reach the hyperplane $\{x_d = 2\}$ before exiting \mathcal{C}_1 , and remain in the region $\{x_d > 1\}$ for a unit time with high probability.

Suppose that the exit probability estimate (4.9) is true. Then, taking into account the uniqueness assumption, $\{\mathbb{P}_x\}$ forms a strong Markov process (see [50]), we infer that for any $x \in \mathcal{C}_{\kappa,1}$,

$$\begin{aligned} \mathcal{F}(\mathbb{P}_x) &= \mathbb{E}_x \int_0^{\sigma_0} e^{-t} f(\omega_t) dt + \mathbb{E}_x \left[e^{-\sigma_0} \left(\int_0^{\infty} e^{-t} f(\omega_t) dt \right) \circ \theta_{\sigma_0} \right] \\ &\geq \mathbb{E}_x \left[\mathbf{1}_{\{\bar{\tau} < 1 \wedge \tau, \sigma_1 > 1 + \bar{\tau}\}} \int_{\bar{\tau}}^{\sigma_1} e^{-t} dt \right] \\ &\quad + \mathbb{E}_x e^{-\sigma_0} \mathbb{E}_{\omega_{\sigma_0}} \int_0^{\infty} e^{-t} f(\omega_t) dt \geq p_1 (e^{-1} - e^{-2}) =: p_0 > 0, \end{aligned}$$

where θ_t is the shift operator, and we used the fact that $f(x) = 1$ for all x with $x_d \geq 1$ and (4.8), (4.9) in the last step. This leads to the desirable limit (4.6).

Therefore, we are left to show the exit probability estimate (4.9).

For this purpose, we let

$$B_t(\omega) := \omega_t - \omega_0 - \int_0^t b(\omega_s) ds, \quad t \geq 0,$$

which is a Brownian motion under \mathbb{P}_x with variance $2t$, for any $x \in \mathbb{R}^d$. Set

$$\Omega_N := \left\{ \omega \in \Omega : |B_s(\omega) - B_t(\omega)| \leq N^{\frac{1}{2(1+\alpha)}} |s - t|^{\frac{1}{1+\alpha}}, s, t \in [0, 10] \right\}$$

and

$$\Omega^x := \{\omega \in \Omega : \omega_0 = x\} \quad \text{and} \quad \Omega_N^x := \Omega_N \cap \Omega^x, \quad \text{for all } x \in \mathbb{R}^d.$$

Since $0 < 1/(1 + \alpha) < 1/2$, we can choose $N \gg 1$ such that

$$\mathbb{P}_x(\Omega_N^x) \geq 1/2, \quad \text{for all } x \in \mathbb{R}^d.$$

Next we aim to show that for each $x \in \mathcal{C}_{\kappa,1}$, all paths in Ω_N^x reach the hyperplane $\{y \in \mathbb{R}^d : y_d = 2\}$ before time $1 \wedge \tau$ when $N \gg 1$, i.e.

$$\bar{\tau}(\omega) < 1 \wedge \tau(\omega), \quad \text{for all } x \in \mathcal{C}_{\kappa,1} \text{ and } \omega \in \Omega_N^x. \quad (4.10)$$

In other words, all trajectories in Ω_N^x correspond to the green paths depicted in Figure 2.

If (4.10) holds, then

$$\mathbb{P}_x(\bar{\tau} < 1 \wedge \tau) \geq \mathbb{P}_x(\Omega_N^x) \geq 1/2, \quad x \in \mathcal{C}_{\kappa,1}.$$

Since b is uniformly bounded on $\{x \in \mathbb{R}^d : x_d \geq 1\}$, there exists a positive constant $p_2 > 0$ such that

$$\inf_{\{x \in \mathbb{R}^d : x_d = 2\}} \mathbb{P}_x(\sigma_1 > 1) \geq p_2 > 0.$$

Using the above two estimates and the strong Markov property, we thus obtain that for all $x \in \mathcal{C}_{\kappa,1}$,

$$\begin{aligned} \mathbb{P}_x(\bar{\tau} < 1 \wedge \tau, \sigma_1 > 1 + \bar{\tau}) &= \mathbb{P}_x(\bar{\tau} < 1 \wedge \tau, \sigma_1 \circ \theta_{\bar{\tau}} > 1) \\ &\geq \mathbb{P}_x(\bar{\tau} < 1 \wedge \tau) \inf_{\{x \in \mathbb{R}^d : x_d = 2\}} \mathbb{P}_x(\sigma_1 > 1) \\ &\geq p_2/2 =: p_1 > 0, \end{aligned}$$

which yields (4.9), thereby finishing the proof.

Step 4: Trajectories wander within the cone. Now, the proof reduces to proving (4.10). For this purpose, we shall first show in this step that for N large enough,

$$\Omega_N^x \subseteq \{\omega \in \Omega^x : \omega_t \in \mathcal{C}_1, \forall t \in [0, t_x]; \omega_{t_x} \in \mathcal{C}_\kappa\}, \quad x \in \bar{\mathcal{C}}_{\kappa,2} \setminus \{0\}, \quad (4.11)$$

where

$$t_x := N^{-1}|x_d|^{1+\alpha}, \quad x \in \mathbb{R}^d. \quad (4.12)$$

The above inclusion indicates that the solution paths starting from $x \in \bar{\mathcal{C}}_{\kappa,2}$ stay in the larger cone \mathcal{C}_1 over a certain time interval $[0, t_x]$, and return back to the small cone \mathcal{C}_κ at time t_x .

In order to prove (4.11), we define

$$V_t^x(\omega) := x_d + \int_0^t b_d(\omega_s) ds \quad \text{and} \quad \hat{x} := (x_1, \dots, x_{d-1}).$$

Obviously,

$$(\omega_t)_d = V_t^x(\omega) + (B_t(\omega))_d, \quad \text{for all } \omega \in \Omega^x.$$

Then, by the construction of the drift in (4.2) and (4.3), we have that

$$b_d(y) = N(d-1)y_d^{-\alpha}, \quad \text{for all } y \in \mathcal{C}_1$$

and for $1 \leq i \leq d-1$,

$$b_i(y) = N\alpha(y_i y_d^{-\alpha-1}), \quad \text{for all } y \in \mathcal{C}_1.$$

Thus, for all $x \in \mathcal{C}_1, \omega \in \Omega^x$ and $t \in [0, \tau(\omega)]$,

$$\begin{aligned} V_t^x(\omega) - x_d &= \int_0^t b_d(\omega_s) ds = N(d-1) \int_0^t [V_s^x(\omega) + (B_s(\omega))_d]^{-\alpha} ds \geq 0, \\ \hat{\omega}_t - \hat{x} &= N\alpha \int_0^t [V_s^x(\omega) + (B_s(\omega))_d]^{-\alpha-1} \hat{\omega}_s ds + \hat{B}_t(\omega). \end{aligned} \quad (4.13)$$

Then by (4.13) and (4.12), we have

$$\begin{aligned} |B_t(\omega) - B_0(\omega)| &= |B_t(\omega)| \leq N^{\frac{1}{2(1+\alpha)}} t^{\frac{1}{1+\alpha}} \\ &= \varepsilon(N)x_d \leq \varepsilon(N)V_t^x(\omega), \quad \text{for all } x \in \bar{\mathcal{C}}_{\kappa,2} \setminus \{0\}, \omega \in \Omega_N^x \text{ and } t \in [0, t_x \wedge \tau(\omega)], \end{aligned} \quad (4.14)$$

where

$$\varepsilon = \varepsilon(N) := N^{-\frac{1}{2(1+\alpha)}} \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

This together with (4.13) yield that

$$(1 + \varepsilon)^{-\alpha} N(d-1)(V_t^x(\omega))^{-\alpha} \leq \frac{dV_t^x(\omega)}{dt} \leq (1 - \varepsilon)^{-\alpha} N(d-1)(V_t^x(\omega))^{-\alpha},$$

for all $x \in \bar{\mathcal{C}}_{\kappa,2} \setminus \{0\}$, $\omega \in \Omega_N^x$ and $t \in [0, t_x \wedge \tau(\omega)]$. By virtue of Chaplygin's Lemma, we derive

$$\begin{aligned} \underline{V}_t^x(\omega) &:= [x_d^{1+\alpha} + (1 + \varepsilon)^{-\alpha} N(d-1)(\alpha+1)t]^{\frac{1}{1+\alpha}} \\ &\leq V_t^x(\omega) \leq [x_d^{1+\alpha} + (1 - \varepsilon)^{-\alpha} N(d-1)(\alpha+1)t]^{\frac{1}{1+\alpha}} =: \bar{V}_t^x(\omega), \end{aligned} \quad (4.15)$$

for all $x \in \bar{\mathcal{C}}_{\kappa,2} \setminus \{0\}$, $\omega \in \Omega_N^x$ and $t \in [0, t_x \wedge \tau(\omega)]$. This along with (4.14) yields that

$$(\omega_t)_d = V_t^x(\omega) + (B_t(\omega))_d \geq (1 - \varepsilon)V_t^x(\omega) \geq (1 - \varepsilon)\underline{V}_t^x(\omega), \quad (4.16)$$

for all $x \in \bar{\mathcal{C}}_{\kappa,2} \setminus \{0\}$, $\omega \in \Omega_N^x$ and $t \in [0, t_x \wedge \tau(\omega)]$.

Next we aim to show that

$$t_x < \tau(\omega), \quad \text{for all } x \in \bar{\mathcal{C}}_{\kappa,2} \setminus \{0\} \text{ and } \omega \in \Omega_N^x. \quad (4.17)$$

Actually, since $1 < \kappa < (d-1)/\alpha$, $|\hat{x}| < x_d/\kappa$ for each $x \in \mathcal{C}_\kappa$, it follows from (4.13) and (4.14) that

$$\begin{aligned} |\hat{\omega}_t| &\leq |\hat{x}| + N\alpha \left| \int_0^t \hat{\omega}_s [V_s^x(\omega) + (B_s(\omega))_d]^{-\alpha-1} ds \right| + |\hat{B}_t(\omega)| \\ &\leq |\hat{x}| + \varepsilon V_t^x(\omega) + N\alpha \int_0^t [V_s^x(\omega) + (B_s(\omega))_d]^{-\alpha} ds \\ &= |\hat{x}| + \varepsilon V_t^x(\omega) + \frac{\alpha}{d-1} (V_t^x(\omega) - x_d) \\ &\leq \left(\varepsilon + \frac{\alpha}{d-1} \right) V_t^x(\omega) + \left(\frac{1}{\kappa} - \frac{\alpha}{d-1} \right) x_d \\ &\leq \left(\varepsilon + \frac{1}{\kappa} \right) V_t^x(\omega), \quad \text{for all } x \in \bar{\mathcal{C}}_{\kappa,2} \setminus \{0\}, \omega \in \Omega_N^x \text{ and } t \in [0, t_x \wedge \tau(\omega)], \end{aligned} \quad (4.18)$$

where the last step is due to the fact that $x_d \leq V_t^x$ implied by (4.13). In the second inequality above, we also used the fact that $|\hat{\omega}_t| \leq (\omega_t)_d$ when $t \in [0, \tau(\omega)]$. Combining the above estimate with (4.15) and (4.16), we come to

$$\frac{(\omega_t)_d}{|\hat{\omega}_t|} \geq (1 - \varepsilon) \left(\varepsilon + \frac{1}{\kappa} \right)^{-1} > 1, \quad \text{for all } x \in \bar{\mathcal{C}}_{\kappa,2} \setminus \{0\}, t \in [0, t_x \wedge \tau(\omega)] \text{ and } \omega \in \Omega_N^x,$$

provided that N is sufficiently large. Since $\tau(\omega)$ is the first exit time of ω_t from \mathcal{C}_1 , we get $t_x \wedge \tau(\omega) < \tau(\omega)$, which is (4.17).

Moreover, via (4.15), for sufficiently large N , $x \in \bar{\mathcal{C}}_{\kappa,2} \setminus \{0\}$ and $\omega \in \Omega_N^x$, we also have

$$\bar{V}_{t_x}^x(\omega) \geq V_{t_x}^x(\omega) \geq \underline{V}_{t_x}^x(\omega) \geq 2^{1/3} x_d,$$

and so by (4.16)-(4.18),

$$\begin{aligned} \frac{(\omega_{t_x})_d}{|\widehat{\omega}_{t_x}|} &\geq \frac{(1-\varepsilon)V_{t_x}^x(\omega)}{\left(\varepsilon + \frac{\alpha}{d-1}\right)\overline{V}_{t_x}^x(\omega) + \left(\frac{1}{\kappa} - \frac{\alpha}{d-1}\right)x_d} \\ &\geq \frac{(1-\varepsilon)V_{t_x}^x(\omega)}{\left[\varepsilon + 2^{-1/3}\kappa^{-1} + (1-2^{-1/3})\frac{\alpha}{d-1}\right]\overline{V}_{t_x}^x(\omega)} > \kappa, \end{aligned}$$

where the last inequality was also due to (4.12), (4.15) and the fact that $\kappa < (d-1)/\alpha$ and that $\varepsilon \rightarrow 0$ as $N \rightarrow \infty$. Thus,

$$\omega_{t_x} \in \mathcal{C}_\kappa, \text{ for any } x \in \overline{\mathcal{C}}_{\kappa,2} \setminus \{0\} \text{ and } \omega \in \Omega_N^x,$$

provided that $N \gg 1$.

To sum up, we obtain the desired inclusion (4.11) for N large enough.

Step 5: Proof of exit probability estimate. Now we are ready to prove (4.10), and so the exit probability estimate (4.9).

We begin by showing that

$$\overline{\tau} < 1, \quad \text{for all } x \in \mathcal{C}_{\kappa,1} \text{ and } \omega \in \Omega_N^x. \quad (4.19)$$

If not, then there exist $x \in \mathcal{C}_{\kappa,1}$ and $\omega \in \Omega_N^x$ such that $(\omega_{1/2})_d < 2$. On the other hand, using that $b_d(y) = N(d-1)y_d^{-\alpha}$, $y \in \mathcal{C}_{1,2}$ and (4.13), we have

$$(\omega_{1/2})_d = V_{1/2}^x(\omega) + (B_{1/2}(\omega))_d \geq x_d + N(d-1)2^{-\alpha-1} - N^{\frac{1}{2(1+\alpha)}} > 2.$$

This contradiction implies (4.19).

So, it remains to show that $\overline{\tau}(\omega) < \tau(\omega)$ when $x \in \mathcal{C}_{\kappa,1}$, $\omega \in \Omega_N^x$ (this implies the red path depicted in Figure 2 cannot belong to Ω_N^x). Suppose, for contradiction, that this is not the case. Then there exist $x \in \mathcal{C}_{\kappa,1}$ and $\omega \in \Omega_N^x$ such that $\overline{\tau}(\omega) = \tau(\omega)$ and $\omega_{\overline{\tau}} \in \partial\mathcal{C}_1$. Let $S(\omega)$ denote the last time at which ω exits $\mathcal{C}_{\kappa,2}$ before $\overline{\tau}(\omega)$, i.e.

$$S(\omega) = \sup \{0 < t < \overline{\tau}(\omega) : \omega_t \notin \mathcal{C}_{\kappa,2}\}.$$

Then $S(\omega) < \overline{\tau}(\omega) < 1$, $\omega_S \in \partial\mathcal{C}_{\kappa,2} \setminus \{0\}$ and $t_{\omega_S} < N^{-1}2^3 < 9$. Therefore, the inclusion in (4.11) for $x \in \overline{\mathcal{C}}_{\kappa,2} \setminus \{0\}$ and $\omega \in \Omega_N^x$ also holds, respectively, for the new starting point ω_S and the new path $\theta_S\omega$. Thus, we have

$$S(\omega) < S(\omega) + t_{S(\omega)} < \tau(\omega) = \overline{\tau}(\omega), \quad \omega_{S+t_S} \in \mathcal{C}_{\kappa,2},$$

which however contradicts the definition of S . Finally, we obtain (4.10) and finish the proof. \square

4.2. FPE with supercritical drifts. We close this section with the proof for the non-uniqueness of linear Fokker-Planck equations with supercritical drifts.

Proof of Corollary 1.4. Let b be the same drift constructed in the above proof of Theorem 1.3. By [54, Theorem 1.1], there exists at least one weak solution to (SDE) satisfying the Aronson-type estimate (KE) for any $x \in \mathbb{R}^d$, and the one-dimensional marginal distribution of this solution solves (3.5) with the initial data δ_x .

Suppose that there is only one solution ρ to (3.5) with $\rho(0) = \delta_0$ in $C_t\mathcal{P}_x$. For any $x \in \mathbb{R}^d$, let \mathbb{P}_x be any martingale solution to (SDE) with $s = 0$. We shall prove that the one-dimensional marginal distribution of \mathbb{P}_x is unique.

For this purpose, let $\sigma = \inf\{t > 0 : \omega_t = 0\}$. Because the drift b is smooth in $\mathbb{R}^d \setminus \{0\}$ and bounded on $B_\varepsilon^c(0)$ for any $\varepsilon > 0$, by the standard SDE well-posedness theory, for any $x \neq 0$ away from the origin, the restriction $\mathbb{P}_x \upharpoonright_{\mathcal{F}_\sigma}$ of the martingale solution \mathbb{P}_x to the filtration \mathcal{F}_σ is uniquely determined. Then, for any $x \neq 0$ and any bounded measurable function f ,

$$\begin{aligned} \mathbb{E}_x(f(\omega_t)) &= \mathbb{E}_x(f(\omega_t)\mathbf{1}_{\{\sigma \leq t\}}) + \mathbb{E}_x(f(\omega_t)\mathbf{1}_{\{\sigma > t\}}) \\ &= \mathbb{E}_x[\mathbf{1}_{\{\sigma \leq t\}}\mathbb{E}_x(f(\omega_t)|\mathcal{F}_\sigma)] + \mathbb{E}_x(f(\omega_{t \wedge \sigma})\mathbf{1}_{\{\sigma > t\}}) \\ &= \mathbb{E}_x[\langle f, \rho_\sigma(t - \sigma) \rangle \mathbf{1}_{\{\sigma \leq t\}}] + \mathbb{E}_x(f(\omega_{t \wedge \sigma})\mathbf{1}_{\{\sigma > t\}}), \end{aligned} \quad (4.20)$$

where the last identity was due to the uniqueness assumption and the fact that the regular conditional probability $\mathbb{P}_x(\cdot|\mathcal{F}_\sigma)(\omega')$ is also a martingale solution to (SDE) starting from 0, for \mathbb{P}_x -a.s. ω' . Taking into account that the distribution of $(\sigma, \omega_{\cdot \wedge \sigma})$ is uniquely determined under \mathbb{P}_x , we thus infer that any one-dimensional marginal distribution of weak solutions to (SDE) satisfies the same identity (4.20), and thus, is unique for all $x \in \mathbb{R}^d$.

Consequently, in view of [50, Theorem 6.2.3], we infer that \mathbb{P}_x is uniquely determined for every $x \in \mathbb{R}^d$, which, however, contradicts the non-uniqueness result in Theorem 1.3. \square

5. MVE AND NFPE WITH CRITICAL SINGULAR KERNELS

5.1. Existence. Let us start with the existence part in Theorem 1.6 (i).

Proof of Theorem 1.6 (i). Let

$$K_1 := K\mathbf{1}_{\{|K|>1\}}, \quad K_2 := K\mathbf{1}_{\{|K|\leq 1\}} \quad \text{and} \quad K^n := K * \varrho^n, \quad K_i^n := K_i * \varrho^n,$$

where ϱ^n is the mollifier as in the proof of Theorem 1.2.

Consider the approximate McKean-Vlasov equation

$$\begin{cases} dX_t^n = K^n(t, X_t^n - y)\rho^n(t)(dy) + \sqrt{2}dW_t \\ \rho^n(t) = \text{law}(X_t^n), \quad \rho^n(0) = \zeta \in \mathcal{P}(\mathbb{R}^d). \end{cases}$$

Its well-posedness is standard, see, for instance, [47, 55]. Set

$$b^n(t, x) := \int_{\mathbb{R}^d} K^n(t, x - y)\rho^n(t, dy) \quad \text{and} \quad b_i^n(t, x) := \int_{\mathbb{R}^d} K_i^n(t, x - y)\rho^n(t, dy), \quad (5.1)$$

where $i = 1, 2$. Then

$$\text{div} b^n = 0, \quad \sup_n \|b^n\|_{L_t^\infty L_x^{d, \infty}} \leq C\|K^n\|_{L_t^\infty L_x^{d, \infty}} \leq C\|K\|_{L_t^\infty L_x^{d, \infty}} < \infty.$$

Arguing as in the proof of Theorem 1.2 and applying the Skorokhold representation theorem we infer that there exist a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a sequence of random maps (Y, B) and $\{(Y^n, B^n)\}$ such that, B^n, B are Brownian motions,

$$(X^n, W^n) \stackrel{d}{=} (Y^n, B^n), \quad (X, W) \stackrel{d}{=} (Y, B), \quad \text{and} \quad (Y^n, B^n) \rightarrow (Y, B), \quad \mathbf{P} - a.s..$$

Moreover, for any $(p, q) \in \mathcal{I}$, the Krylov estimate holds

$$\mathbf{E} \int_0^T f(t, Y_t^n) dt, \quad \mathbf{E} \int_0^T f(t, Y_t) dt \leq C\|f\|_{L_t^q L_x^p}. \quad (5.2)$$

In order to prove that the limit (Y, B) is a weak solution of (MVE), as in the arguments below (3.4) in the proof of Theorem 1.2, it suffices to show that

$$b_i^n \rightarrow b_i := K_i * \rho \quad \text{in} \quad L_t^{q_i} L_x^{p_i}, \quad i = 1, 2, \quad (5.3)$$

where $\rho(t)$ is the distribution of Y_t and $(p_i, q_i) \in \mathcal{I}$ is as in Subsection 3.1.

For this purpose, given a large number $N \in \mathbb{N}$, by the triangle inequality

$$\begin{aligned} |b_1^n - b_1| &\leq |(K_1 - K_1^N) * \rho| + |(K_1^n - K_1^N) * \rho^n| + |K_1^N * \rho - K_1^N * \rho^n| \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Noting that $K_1 \in L_t^\infty L_x^{p_1}$, we have

$$\lim_{N \rightarrow \infty} \|I_1\|_{L_t^{q_1} L_x^{p_1}} \leq \lim_{N \rightarrow \infty} \|K_1 - K_1^N\|_{L_t^{q_1} L_x^{p_1}} = 0$$

and

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \|I_2\|_{L_t^{q_1} L_x^{p_1}} \leq \lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \|K_1^n - K_1^N\|_{L_t^{q_1} L_x^{p_1}} = 0.$$

Regarding I_3 , since $K_1^N \in C_b^\infty(\mathbb{R}^{d+1})$ and $\sup_{t \in [0, T]} |Y_t - Y_t^n| \rightarrow 0$, \mathbf{P} -a.s, it holds that

$$\begin{aligned} &K_1^N * \rho(t, x) - K_1^N * \rho^n(t, x) \\ &= \mathbf{E}[K_1^N(t, x - Y_t) - K_1^N(t, x - Y_t^n)] \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad t \in [0, T], x \in \mathbb{R}^d. \end{aligned} \quad (5.4)$$

Moreover, thanks to (AE), we have

$$|K_1^N * \rho^n|(t, x) \leq C \int_{\mathbb{R}^d} |K_1^N(x - y)| dy \int_{\mathbb{R}^d} h(Ct, z - y) \zeta(dz) =: F^N(t, x). \quad (5.5)$$

Note that

$$\begin{aligned} \|F^N\|_{L_t^{q_1} L_x^{p_1}} &\leq C \|K_1^N\|_{L_t^{q_1} L_x^{p_1}} \|h(Ct) * \zeta\|_{L_t^\infty L_x^1} \leq C \|K_1\|_{L_t^\infty L_x^{p_1}} \\ &\leq C \|K\|_{L_t^\infty L_x^{d, \infty}}^{\frac{d}{p_1}} \left(\int_1^\infty \lambda^{p_1 - 1 - d} d\lambda \right)^{\frac{1}{p_1}} \leq C, \end{aligned}$$

which along with (5.4), (5.5) and the dominated convergence theorem implies that

$$\lim_{n \rightarrow \infty} \|I_3\|_{L_t^{q_1} L_x^{p_1}} = 0.$$

Combining the above estimates for I_i , $i=1,2,3$, we obtain

$$\lim_{n \rightarrow \infty} \|b_1^n - b_1\|_{L_t^{q_1} L_x^{p_1}} \leq \lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^3 \|I_i\|_{L_t^{q_1} L_x^{p_1}} = 0.$$

Similar arguments also apply to the b_2^n component:

$$\lim_{n \rightarrow \infty} \|b_2^n - b_2\|_{L_t^{q_2} L_x^{p_2}} = 0.$$

Thus, we obtain the desired limit (5.3) and finish the proof of the existence part.

Now, let X be the solution to (MVE) obtained by the above approximation procedure. Note that, the drift satisfies the condition (A_b), and thus, by Theorem 1.1, for each $t \in (0, T]$ the distribution of X_t has the density (still denoted by $\rho(t)$) satisfying

$$\rho(t, y) = \int_{\mathbb{R}^d} p_{t_0, t}^b(x, y) \rho(t_0, x) dx, \quad t_0 \in (0, t),$$

where the drift $b(t, x) = K * \rho(t, x)$, and $p_{s, t}^b$ is the transition density corresponding to the backward Kolmogorov operator $\partial_t + \Delta + b \cdot \nabla$, which satisfies the Aronson estimate (AE).

Furthermore, the drift is bounded on positive time, that is,

$$\|b\|_{L^\infty([t_0, T] \times \mathbb{R}^d)} \leq C t_0^{-\frac{1}{2}} \|K\|_{L_t^\infty L_x^{d, \infty}} < \infty, \quad t_0 \in (0, T).$$

Since for each $t_0 \in (0, t)$, $p_{t_0, t}^b = p_{t_0, t}^{b\mathbf{1}_{[t_0, T]}}$ and $b\mathbf{1}_{[t_0, T]}$ is bounded, in view of [11, Theorem 2.3], we get

$$|\nabla \rho(t, y)| \leq \left| \int_{\mathbb{R}^d} \nabla_y p_{t_0, t}^{b\mathbf{1}_{[t_0, T]}}(x, y) \rho_{t_0}(dx) \right| \leq C(d, t_0, K) |t - t_0|^{-\frac{1}{2}}, \quad t_0 < t \leq T.$$

An application of inductive arguments then leads to

$$\|\nabla^k \rho(t)\|_{L_x^\infty} \leq C(k, t) < \infty, \quad k \geq 0, \quad t \in (0, T].$$

This gives the regularity of the corresponding density. \square

Proof of Theorem 1.7(i). Let K^n be the same kernel as in the proof of Theorem 1.6, and let ρ^n be the solution to (NFPE) with K replaced by K^n . Let b^n be given by (5.1) and p^n the transition density function associated with the operator $\Delta + b^n \cdot \nabla$.

We first claim that (ρ^n) is relatively compact in $C_t \mathcal{M}_x$. To this end, we note that

$$\|\rho^n(t)\|_{\mathcal{M}} \leq C \left\| \int_{\mathbb{R}^d} p_{0, t}^n(x, \cdot) |\zeta|(dx) \right\|_{\mathcal{M}} \leq \|\zeta\|_{\mathcal{M}}, \quad t \in [0, T],$$

which implies that $(\rho^n(t))$ is compact in \mathcal{M} , for each $t \in [0, T]$. According to the Arzelà-Ascoli theorem, we only need to prove that (ρ^n) is equicontinuous in $C_t \mathcal{M}_x$. For this purpose, taking any $f \in C_0(\mathbb{R}^d)$, we have

$$\begin{aligned} \langle \rho_{t+\delta}^n - \rho^n(t), f \rangle &= \iint_{\mathbb{R}^{2d}} [p_{0, t+\delta}^n(x, y) - p_{0, t}^n(x, y)] f(y) \zeta(dx) dy \\ &= \int_{\mathbb{R}} \zeta(dx) \int_{\mathbb{R}^d} p_{0, t}^n(x, y) \left(\int_{\mathbb{R}^d} p_{t, t+\delta}^n(y, z) f(z) dz - f(y) \right) dy. \end{aligned}$$

Since

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} p_{t, t+\delta}^n(y, z) f(z) dz - f(y) \right| \\ & \leq \int_{|z-y| < \sqrt[3]{\delta}} |f(y) - f(z)| p_{t, t+\delta}^n(y, z) dz + 2 \|f\|_{L^\infty} \int_{|z-y| > \sqrt[3]{\delta}} p_{t, t+\delta}^n(y, z) dz \\ & \leq C \int_{|z-y| < \sqrt[3]{\delta}} |f(y) - f(z)| h(C\delta, y-z) dz + C \|f\|_{L^\infty} \int_{|z-y| > \sqrt[3]{\delta}} h(C\delta, y-z) dz \\ & \leq C \operatorname{Osc}_{B_{\sqrt[3]{\delta}}(y)} f + C\delta, \end{aligned}$$

We get

$$\sup_n |\langle \rho_{t+\delta}^n - \rho^n(t), f \rangle| \leq C \sup_{y \in \mathbb{R}^d} \operatorname{Osc}_{B_{\sqrt[3]{\delta}}(y)} f + C\delta \rightarrow 0,$$

which yields the equicontinuity of (ρ^n) , as claimed.

Now, let ρ be the limit of (ρ^n) along a subsequence, which, for simplicity, we still denote by n . Noting that ρ^n is bounded in $L_t^q L_x^p$, for any $(p, q) \in \mathcal{I}$, therefore, we also have $\rho^n \rightharpoonup \rho$ in $L_t^q L_x^p$. Then, as in the proof of (5.3), one can verify that

$$\int_0^t \langle \rho^n(s), K^n * \rho^n(s) \cdot \nabla \phi(s) \rangle ds \rightarrow \int_0^t \langle \rho(s), K * \rho(s) \cdot \nabla \phi(s) \rangle ds, \quad \phi \in C_c^\infty([0, T] \times \mathbb{R}^d),$$

which implies that ρ is a distributional solution to (NFPE). \square

5.2. Uniqueness. Next, we prove the uniqueness part in Theorem 1.6. Recall that $\zeta \in \mathcal{M}(\mathbb{R}^d)$ can be decomposed uniquely as $\zeta = \zeta_c + \zeta_a$, where ζ_c is the continuous part, i.e., $\zeta_c(\{x\}) = 0$ for all $x \in \mathbb{R}^d$, and $\zeta_a = \sum_i c_i \delta_{x_i}$ is the purely atomic part. Also recall that h is the Gaussian kernel given by (1.2).

We shall use the following estimate for Gaussian kernel proved by Giga-Miyakawa-Osada [21].

Lemma 5.1 ([21]). *For any $r > 1$ and $\beta \geq 1$,*

$$\limsup_{t \rightarrow 0} t^{\frac{d}{2r'}} \|h(t) * \zeta\|_{L_x^{r,\beta}} \leq C_7(d, r) |\zeta_a|,$$

where $|\zeta_a|$ is the total variation of ζ_a of \mathbb{R}^d , and r' is the conjugate number of r .

Proof of the uniqueness part in Theorem 1.6: Suppose that $Y^{(1)}$ and $Y^{(2)}$ are two weak solutions to (MVE) with the density $\rho^{(1)}$ and $\rho^{(2)}$, respectively, and satisfy the Krylov-type estimate (1.3). Put $b^{(i)} = K * \rho^{(i)}$, $i = 1, 2$. Since

$$\|b^{(i)}\|_{L_t^\infty L_x^{d,\infty}} \leq C(d) \|K\|_{L_t^\infty L_x^{d,\infty}},$$

thanks to Theorem 1.2, we have the representation formula of solutions

$$\rho^{(i)}(t, y) = \int_{\mathbb{R}^d} p_{0,t}^{b^{(i)}}(x, y) \zeta(x) dx, \quad i = 1, 2, \quad (5.6)$$

where $p_{0,t}^{b^{(i)}}$ is the transition density of the unique solution to (SDE) with the drift $b = b^{(i)}$ starting from the initial time $s = 0$. Moreover, by (5.6), the Aronson-type estimate (AE) and Lemma 5.1, there exists $T_0 > 0$ such that

$$\sup_{t \in [0, T_0]} t^{\frac{d}{2r'}} \|\rho^{(i)}(t)\|_{L_x^{r,\beta}} \leq C_8 |\zeta_a|, \quad \forall r > 1, \beta \geq 1. \quad (5.7)$$

Set

$$\rho(t) := \rho^{(1)}(t) - \rho^{(2)}(t) \quad \text{and} \quad b := b^{(1)} - b^{(2)}.$$

It follows from the Duhamel formula of $\rho^{(i)}$, $i = 1, 2$, that

$$\begin{aligned} \rho(t) &= \int_0^t \nabla \cdot h(t-s) * (b^{(1)}(s) \rho(s)) ds - \int_0^t \nabla \cdot h(t-s) * (b(s) \rho^{(2)}(s)) ds \\ &=: J_1(t) - J_2(t). \end{aligned} \quad (5.8)$$

In light of Lemma 5.1, we set the following norm as in [21]

$$\|f\|_{r, T_0} := \sup_{t \in [0, T_0]} t^{\frac{d}{2r'}} \|f(t)\|_{L_x^r}$$

for any $r \in (1, d')$, where r' is the conjugate exponent of r , which measures the propagation of heat flows.

By (5.7) and Young's inequality (A.4), for any $s \in [0, T_0]$,

$$\begin{aligned} \|b^{(1)}(s) \rho(s)\|_{L_x^r} &\leq \|b^{(1)}(s)\|_{L_x^\infty} \|\rho(s)\|_{L_x^r} \leq C \|\rho^{(1)}(s)\|_{L_x^{d',1}} \|\rho(s)\|_{L_x^r} \\ &\leq C |\zeta_a| s^{-\frac{1}{2}} \|\rho(s)\|_{L_x^r}. \end{aligned}$$

This yields that

$$\begin{aligned} \|J_1\|_{r, T_0} &\leq C |\zeta_a| \sup_{t \in [0, T]} t^{\frac{d}{2r'}} \left[\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}(1+\frac{d}{r'})} \|\rho\|_{r, T_0} ds \right] \\ &\leq CB \left(\frac{1}{2}, \frac{1}{2} \left(1 - \frac{d}{r'} \right) \right) |\zeta_a| \|\rho\|_{r, T_0}. \end{aligned} \quad (5.9)$$

Regarding the second term J_2 , using (5.7) and Young's inequality (A.3) again we derive

$$\|b(s)\rho^{(2)}(s)\|_{L_x^r} \leq \|b(s)\|_{L_x^l} \|\rho^{(2)}(s)\|_{L_x^{d'}} \leq C|\zeta_a|s^{-\frac{1}{2}}\|\rho(s)\|_{L_x^r}$$

with $r \in (1, d')$ and $1/l = 1/r + 1/d - 1$. Thus, we have

$$\begin{aligned} \|J_2\|_{r, T_0} &\leq |\zeta_a| \sup_{t \in [0, T]} t^{\frac{d}{2r'}} \left[\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}(1+\frac{d}{r'})} \|\rho\|_{r, T_0} ds \right] \\ &\leq CB \left(\frac{1}{2}, \frac{1}{2} \left(1 - \frac{d}{r'} \right) \right) |\zeta_a| \|\rho\|_{r, T_0}. \end{aligned} \quad (5.10)$$

Therefore, plugging (5.9) and (5.10) into (5.8) and using the smallness of the initial mass we thus get

$$\|\rho\|_{r, T_0} \leq C_9 |\zeta_a| \|\rho\|_{r, T_0} \leq \varepsilon_0 C_9 \|\rho\|_{r, T_0},$$

where C_9 only depends on d, r and $\|K\|_{L_t^\infty L_x^{d, \infty}}$. It follows that $\|\rho\|_{r, T_0} = 0$ if $\varepsilon_0 < C_9^{-1}$.

Note that for any $\rho \in C_t \mathcal{P}_x$, the convolution $K * \rho$ belongs to the critical Lorentz space $L_t^\infty L_x^{d, \infty}$ and satisfies the divergence free condition $\operatorname{div}(K * \rho) = 0$. In view of the uniqueness result for (SDE) in Theorem 1.2, for solutions in the Krylov class, the uniqueness of the path law for (MVE) is a consequence of the uniqueness of the marginal distributions.

Therefore, the proof of Theorem 1.6 is complete. \square

The uniqueness part of Theorem 1.7 can be proved in an analogous manner as that of Theorem 1.6, with the modification that Proposition 3.2 is used in place of Theorem 1.1. So the details are omitted here.

Next, we present the proof for Theorem 1.8.

Proof of Theorem 1.8. The uniqueness with smallness condition follows immediately from Theorem 1.7. In the case without smallness condition, by virtue of Proposition 3.2, for any distributional solution ρ to (NSE) with $\rho(0) = \zeta \in \mathcal{M}(\mathbb{R}^2)$ and satisfying (1.4), one still has the following representation formula:

$$\rho(t, y) = \int_{\mathbb{R}^2} p_{0,t}^b(x, y) \zeta(x) dx, \quad (5.11)$$

where $b = K_{\text{BS}} * \rho$, and $p_{0,t}^b$ is the transition density of the unique solution to (SDE) with $s = 0$. Using this representation formula and the Aronson-type estimate (AE) we derive

$$\|\rho(t)\|_{L^\infty} \leq C \|h(Ct, \cdot)\|_{L^\infty} |\mu| \leq \frac{C}{t} |\zeta|, \quad t \in (0, T]$$

and

$$\|\rho(t)\|_{L^1} \leq C \|h(Ct, \cdot)\|_{L^1} |\zeta| \leq C |\zeta| < \infty, \quad t \in [0, T].$$

Moreover, for each $t > 0$, (2.2) implies that $\rho(t) \in C_0(\mathbb{R}^2)$. Thus, we infer that $\rho \in C((0, T]; L^1 \cap L^\infty)$ and the solution lives in the uniqueness solution class in [18]. Therefore, by virtue of [18, Theorem 1.1], we obtain the desired uniqueness assertion. \square

Our result for the 2D vorticity NSE also implies the uniqueness for the original 2D NSE in the velocity formulation. This is the content of Corollary 5.2 below.

Corollary 5.2. *Let $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{S}$ be the pair as in Theorem 1.8. Suppose that $u^{(1)}$ and $u^{(2)}$ are two distributional solutions to the 2D NSE*

$$\partial_t u + (u \cdot \nabla) u = \Delta u - \nabla p, \quad \operatorname{div} u = 0, \quad u|_{t=0} = u_0.$$

and that $u^{(i)} \in L_t^{q'} \dot{W}_x^{1,p'}$, $u^{(i)}(t) \in L_x^{2,\infty}$ and $\nabla \times u^{(i)} := \partial_1 u_2^{(i)} - \partial_2 u_1^{(i)} \in C_t \mathcal{M}_x$, $i = 1, 2$. Then $u^{(1)} = u^{(2)}$.

We need the following auxiliary lemma.

Lemma 5.3. *Suppose that F is a divergence-free vector field in $L^{2,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ and $\nabla \times F \in \mathcal{M}$. Then*

$$F = K_{\text{BS}} * (\nabla \times F),$$

where K_{BS} is the Biot-Savart law given by (1.1).

Proof. Let $G := K_{\text{BS}} * (\nabla \times F) \in L^{2,\infty}$ and $H := F - G$. Then, $\text{div} H = 0$ and $\nabla \times H = 0$ in the sense of distribution. This implies

$$\begin{aligned} 0 &= \partial_1(\text{div} H) - \partial_2(\nabla \times H) = \Delta H_1, \\ 0 &= \partial_2(\text{div} H) + \partial_1(\nabla \times H) = \Delta H_2 \end{aligned}$$

in the sense of distribution. Noting that $H \in L^{2,\infty} \subseteq L_{loc}^1$, by the mean value theorem for harmonic functions, for each $x \in \mathbb{R}^2$, we derive

$$\begin{aligned} |H(x)| &= \frac{1}{|B_R|} \left| \int_{B_R(x)} H(y) dy \right| \leq CR^{-2} \int_0^\infty |\{y \in B_R(x) : |H(y)| > t\}| dt \\ &\leq CR^{-2} \int_0^\infty (t^{-2} \wedge R^2) dt \leq CR^{-1} \rightarrow 0 \quad \text{as } R \rightarrow \infty, \end{aligned}$$

which means $F = G = K_{\text{BS}} * (\nabla \times F)$. \square

Proof of Corollary 5.2. We omit the superscript i below for simplicity. Let

$$\rho := \nabla \times u \in L_t^{q'} L_x^{p'} \cap C_t \mathcal{M}_x.$$

In view of Lemma 5.3, we have $u = K_{\text{BS}} * \rho$, where K_{BS} is the Biot-Savart law given by (1.1).

For any $\phi \in C_c^\infty([0, T] \times \mathbb{R}^2)$, set $\varphi = \nabla^\perp \phi$. Then φ is divergence-free. The integration-by-parts formula yields that

$$\langle \rho, \phi \rangle = \langle \nabla \times K_{\text{BS}} * \rho, \phi \rangle = -\langle u, \nabla^\perp \phi \rangle = -\langle u, \varphi \rangle$$

and

$$\langle u \rho, \nabla \phi \rangle = \langle \nabla \times u, u \cdot \nabla \phi \rangle = -\langle u, \nabla^\perp (u \cdot \nabla \phi) \rangle = -\int u_i u_j \partial_j \varphi_i.$$

Moreover, by the equation of u , for almost every $s, t \in (0, T)$,

$$\begin{aligned} &\langle u(t), \varphi(t) \rangle - \langle u(s), \varphi(s) \rangle \\ &= \int_s^t \langle u(r), \partial_r \varphi(r) + \Delta \varphi(r) \rangle dr + \int_s^t dr \int_{\mathbb{R}^2} u_i(r) u_j(r) \partial_j \varphi_i(r). \end{aligned}$$

Thus, combining the above calculations together we derive

$$\langle \rho(t), \phi(t) \rangle - \langle \rho(s), \phi(s) \rangle = \int_s^t \langle \rho(r), \partial_r \phi(r) + \Delta \phi(r) + u(r) \cdot \nabla \phi(r) \rangle dr,$$

for almost every $s, t \in (0, T)$. Taking into account $\rho \in C_t \mathcal{M}_x$, we then obtain

$$\langle \rho(t), \phi(t) \rangle - \langle \nabla \times u_0, \phi(0) \rangle = \int_0^t \langle \rho(s), \partial_s \phi(s) + \Delta \phi(s) + u(s) \cdot \nabla \phi(s) \rangle ds, \quad t \in [0, T].$$

That is, ρ satisfies the 2D vorticity NSE.

Therefore, by virtue of Theorem 1.8, we obtain $\nabla \times u^{(1)} = \nabla \times u^{(2)}$, which along with Lemma 5.3 yields the desired uniqueness $u^{(1)} = u^{(2)}$. \square

6. NONLINEAR MARKOV PROCESSES

Let us first review the notions from [46]. Let $\Omega_s := C([s, T]; \mathbb{R}^d)$ be the set of all continuous paths in \mathbb{R}^d starting from time s equipped with the topology of locally uniform convergence, and $\mathcal{B}(\Omega_s)$ the corresponding Borel σ -algebra. Define for $0 \leq s \leq r \leq t \leq T$,

$$\begin{aligned} \pi_t^s : \Omega_s &\rightarrow \mathbb{R}^d; \quad \pi_t^s(\omega) = \omega(t), \quad \omega \in \Omega_s, \\ \mathcal{F}_t^s &:= \sigma(\pi_r^s : s \leq r \leq t), \end{aligned}$$

and

$$\theta_r^s : \Omega_s \rightarrow \Omega_r; \quad (\theta_r^s \omega)(t) = \omega(t).$$

Definition 6.1 (Nonlinear Markov Process [46]). *Let $\mathcal{Q} \subseteq \mathcal{P}(\mathbb{R}^d)$. A nonlinear Markov process is a family of probability measures $(\mathbb{P}_{s,\zeta})_{(s,\zeta) \in [0,T] \times \mathcal{Q}}$ on $\mathcal{B}(\Omega_s)$ such that*

- (i) *For all $0 \leq s \leq r \leq t$ and $\zeta \in \mathcal{Q}$, the marginals $\mu_t^{s,\zeta} := \mathbb{P}_{s,\zeta} \circ (\pi_t^s)^{-1}$ belong to \mathcal{Q} .*
- (ii) *The nonlinear Markov property holds, i.e. for all $0 \leq s \leq r \leq t \leq T, \zeta \in \mathcal{Q}$*

$$\mathbb{P}_{s,\zeta}(\pi_t^s \in A \mid \mathcal{F}_r^s)(\cdot) = p_{(s,\zeta),(r,\pi_r^s(\cdot))}(\pi_t^r \in A), \quad \mathbb{P}_{s,\zeta}\text{-a.s. for all } A \in \mathcal{B}(\mathbb{R}^d)$$

where $p_{(s,\zeta),(r,y)}$, $y \in \mathbb{R}^d$, is a regular conditional probability kernel from \mathbb{R}^d to $\mathcal{B}(\Omega_r)$ of $\mathbb{P}_{r,\mu_r^{s,\zeta}}[\cdot \mid \pi_r^r = y]$, $y \in \mathbb{R}^d$.

Proof for Theorem 1.10. We only provide the proof of (i), as the proof of (ii) follows identically.

Fix $s \geq 0$ and $\zeta \in \mathcal{P}(\mathbb{R}^d)$. Let $\mathbb{P}_{s,\zeta}$ be one of the limiting distributions in $\mathcal{P}(\Omega_s)$ of the approximation processes constructed in Section 5 with initial time s . Then $\mathbb{P}_{s,\zeta}$ is a martingale solution to (MVE) (with initial time s). Set

$$\mu_t^{s,\zeta} := \mathbb{P}_{s,\zeta} \circ (\pi_t^s)^{-1}, \quad s \leq t \leq T.$$

Then, by (5.2), we see that

$$\int_s^T \int_{\mathbb{R}^d} f(t, x) \mu_t^{s,\zeta}(dx) dt = \mathbb{E}_{s,\zeta} \int_s^T f(t, \pi_t^s) dt \leq C \|f\|_{L^q(s,T;L_x^p)},$$

for any $(p, q) \in \mathcal{S}$, namely, $(\mu^{s,\zeta})$ belongs to the Krylov class. Thanks to Theorem 1.7, $\mu^{s,\zeta}$ is uniquely determined when $\zeta \in \mathcal{P}_{\varepsilon_0}$.

In order to prove the desired assertions, by virtue of [46, Corollary 3.9], we shall verify the following two conditions:

- (a) $\{\mu^{s,\zeta}\}_{(s,\zeta) \in [0,T] \times \mathcal{P}(\mathbb{R}^d)}$ satisfies the flow property (1.7) with $\mathcal{Q} = \mathcal{P}(\mathbb{R}^d)$;
- (b) $\mu^{s,\zeta}$ is the unique solution to the linearized Fokker-Planck equation

$$\partial_t \rho = \Delta \rho - \operatorname{div}((K * \mu^{s,\zeta}) \rho), \quad \rho|_{t=s} = \zeta, \quad (6.1)$$

that satisfies

$$\rho(t) \leq C \mu_t^{s,\zeta}, \quad s \leq t \leq T, \quad (6.2)$$

where $C > 0$ is a constant.

To this end, we note that for any $r \in [s, T]$, the path $(\mu_t^{s,\zeta})_{t \in [r, T]}$ solves (NFPE) with initial datum $(r, \mu_r^{s,\zeta})$. In view of Theorem 1.7, for any $r \in (s, T]$, $\mu_r^{s,\zeta}$ admits a bounded density which implies that $\mu_r^{s,\zeta} \in \mathcal{P}_{\varepsilon_0}$. Therefore, invoking Theorem 1.7 once again, the flow property (1.7) is verified.

Moreover, let

$$b(t, x) := \int_{\mathbb{R}^d} K(t, x - y) \mu_t^{s,\zeta}(dy), \quad s \leq t \leq T.$$

Then $b \in L^\infty([s, T]; L_x^{d, \infty})$ and $\operatorname{div} b = 0$. Since $(\mu^{s, \zeta})$ lie in the Krylov class, any curve of probability measures satisfying (6.2) also belongs to the Krylov class. Thus, in view of the well-posedness result in Proposition 3.2, the linear Fokker-Planck equation (6.1) admits a unique probabilistic solution satisfying (6.2), which verifies condition (b).

Therefore, the proof is complete. \square

APPENDIX A. LORENTZ SPACES

This section contains several useful properties of Lorentz spaces and corresponding interpolation estimates. For more detailed explanations, we refer to the nice monograph [22].

Definition A.1. *The Lorentz space is the space of complex-valued measurable functions f on a measure space (X, μ) such that the following quasinorm is finite*

$$\|f\|_{L^{p,q}(X,\mu)} = p^{\frac{1}{q}} \left\{ \int_0^\infty t^{q-1} [\mu(\{x : |f(x)| > t\})]^{\frac{q}{p}} dt \right\}^{\frac{1}{q}},$$

where $0 < p, q \leq \infty$. When $q = \infty$, we set

$$\|f\|_{L^{p,\infty}(X,\mu)} = \sup_{t>0} t [\mu(\{x : |f(x)| > t\})]^{\frac{1}{p}}.$$

It is also conventional to set $L^{\infty,\infty}(X, \mu) = L^\infty(X, \mu)$. The space $L^{p,q}$ is contained in $L^{p,r}$ whenever $q < r$.

Proposition A.1. (i) *Assume that $0 < p, q \leq \infty$ and $\theta > 0$, then*

$$\|f^\theta\|_{L^{p,q}} = \|f\|_{L^{\theta p, \theta q}}^\theta. \quad (\text{A.1})$$

(ii) *(Hölder's inequality) Assume that $1 \leq p_1, p, p_2 \leq \infty$ and $1 \leq q_1, q_2 \leq \infty$, then*

$$\|fg\|_{L^{p,q}} \leq C \|f\|_{L^{p_1, q_1}} \|g\|_{L^{p_2, q_2}}, \quad (\text{A.2})$$

where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

(iii) *(Young's inequality) Assume that $1 < p_1, p, p_2 < \infty$, $1 \leq q_1, q_2 \leq \infty$, then*

$$\|f * g\|_{L^{p,q}} \leq C \|f\|_{L^{p_1, q_1}} \|g\|_{L^{p_2, q_2}}, \quad (\text{A.3})$$

where

$$1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2};$$

and if $1 < p_1, p_2 < \infty$, $1 \leq q_1, q_2 \leq \infty$, then

$$\|f * g\|_{L^\infty} \leq C \|f\|_{L^{p_1, q_1}} \|g\|_{L^{p_2, q_2}}, \quad (\text{A.4})$$

where

$$\frac{1}{p_1} + \frac{1}{p_2} = 1 \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{q_2} \geq 1.$$

(iv) *Assume that $1 < p \leq \infty$ and $1 \leq q \leq \infty$, then*

$$\|f\|_{L^{p,q}} \leq \|\mathcal{M}f\|_{L^{p,q}} \leq C \|f\|_{L^{p,q}}, \quad (\text{A.5})$$

where $\mathcal{M}f$ is the Hardy-Littlewood maximal function of f .

Proof. The proofs of (i)-(iii) can be found in [44, 22]. For (iv), since

$$\|\mathcal{M}f\|_{L^{1,\infty}} \leq C\|f\|_{L^1} \quad \text{and} \quad \|\mathcal{M}f\|_{L^\infty} \leq \|f\|_{L^\infty},$$

by the Marcinkiewicz interpolation theorem for Lorentz spaces (see [26]), one has

$$\|\mathcal{M}f\|_{L^{p,q}} \leq C_p\|f\|_{L^{p,q}}$$

for any $1 < p \leq \infty$. □

For any $s \in (0, d)$. The Riesz potential $I_s f$ of a locally integrable function f on \mathbb{R}^d is defined by

$$(I_s f)(x) := K_s * f(x) = \frac{1}{c_{d,s}} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-s}} dy.$$

We have

Lemma A.2 (Boundedness of Riesz potential). *Let $0 < s < d$ and $p \in (1, d/s)$. There exists $C > 0$ such that for any $f \in L^p$,*

$$\|I_s f\|_{L^{q,p}} \leq C\|f\|_{L^p}, \tag{A.6}$$

where $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$.

Proof. Noting that

$$K_s(x) = \frac{1}{c_{d,s}} \frac{1}{|x|^{d-s}} \in L^{\frac{d}{d-s}, \infty}$$

and using Theorem 2.6 of [44] one has

$$\|I_s f\|_{L^{q,p}} \leq C\|K_s\|_{L^{\frac{d}{d-s}, \infty}}\|f\|_{L^p} \leq C\|f\|_{L^p}.$$

□

APPENDIX B. INTERPOLATION INEQUALITIES

Below we collect several useful interpolation estimates in Lorentz spaces.

Lemma B.1. *Let $1 \leq p_1 < p < p_2 < \infty$, $q \geq 1$ and*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

Then

$$\|f\|_{L^{p,q}} \leq C\|f\|_{L^{p_1,\infty}}^\theta \|f\|_{L^{p_2,\infty}}^{1-\theta}. \tag{B.1}$$

Proof. Let $N_1 = \|f\|_{L^{p_1,\infty}}$ and $N_2 = \|f\|_{L^{p_2,\infty}}$. Then

$$|\{f > t\}| \leq \frac{N_1^{p_1}}{t^{p_1}} \quad \text{and} \quad |\{f > t\}| \leq \frac{N_2^{p_2}}{t^{p_2}}.$$

Thus, by definition, we derive

$$\begin{aligned}
\|f\|_{L^{p,q}} &= p^{\frac{1}{q}} \left(\int_0^\infty t^{q-1} |\{|f| > t\}|^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \\
&= p^{\frac{1}{q}} \left(\int_0^\infty t^{q-1} (N_1^{p_1} t^{-p_1} \wedge N_2^{p_2} t^{-p_2})^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \\
&\leq p^{\frac{1}{q}} \left[N_1^{qp_1/p} \int_0^{\left(\frac{N_2^{p_2}}{N_1^{p_1}}\right)^{\frac{1}{p_2-p_1}}} t^{q-1-\frac{qp_1}{p}} dt + N_2^{qp_2/p} \int_{\left(\frac{N_2^{p_2}}{N_1^{p_1}}\right)^{\frac{1}{p_2-p_1}}}^\infty t^{q-1-\frac{qp_2}{p}} dt \right]^{\frac{1}{q}} \\
&\leq C(p, p_1, p_2, q) N_1^\theta N_2^{1-\theta},
\end{aligned}$$

where $\theta = \frac{1/p-1/p_2}{1/p_1-1/p_2}$. \square

Recall that $\Lambda = (-\Delta)^{1/2}$. For any $s \geq 0$, $1 \leq p < \infty$ and $q > 0$, the homogeneous Lorentz-Bessel space is defined by

$$\dot{H}_{p,q}^s := \{f \in \mathcal{S}'(\mathbb{R}^d) : \Lambda^s f \in L^{p,q}\},$$

and

$$\|f\|_{\dot{H}_{p,q}^s} := \|\Lambda^s f\|_{L^{p,q}}.$$

The following fractional Gagliardo-Nirenberg type estimate in Lorentz spaces involving Besov-Hölder norms is useful in the proof of Theorem 2.7.

Proposition B.2 (Fractional Gagliardo-Nirenberg type estimate). *Let $1 < p, p_1 < \infty$, $1 \leq q, q_1 \leq \infty$, $0 < \alpha < \sigma \leq s < \infty$ and*

$$\theta = \frac{\sigma - \alpha}{s - \alpha} = \frac{p_1}{p} = \frac{q_1}{q} \in (0, 1].$$

Then

$$\|\Lambda^\sigma u\|_{L^{p,q}} \leq C \|\Lambda^s u\|_{L^{p_1, q_1}}^\theta \|u\|_{\dot{B}_{\infty, \infty}^\alpha}^{1-\theta}, \quad \forall u \in \dot{H}_{p_1, q_1}^s \cap \dot{B}_{\infty, \infty}^\alpha, \quad (\text{B.2})$$

where $\dot{B}_{\infty, \infty}^\alpha$ is the homogeneous Besov space (see [1]).

Remark B.3. *Proposition B.2 implies that if $j, m \in \mathbb{N}$, $0 < j < m$, and $1 < p_1 < p < \infty$ such that $\alpha = (jp - mp_1)/(p - p_1) \in (0, 1)$, then*

$$\|\nabla^j u\|_{L^p} \leq C \|\nabla^m u\|_{L^{p_1}}^\theta [u]_\alpha^{1-\theta} \quad \text{with } \theta = p_1/p \in (0, 1), \quad (\text{B.3})$$

where $[u]_\alpha$ is the Hölder seminorm of u . Estimate (B.3) was first proved by Nirenberg in [43].

Proof of Proposition B.2. We only need to consider the case that $\sigma < s$, i.e., $\theta \in (0, 1)$. Let $\dot{\Delta}_j$ and $\dot{\Delta}'_j$ denote the homogeneous dyadic blocks (see [1]) given by cutoff functions φ and φ' , respectively, and $\varphi = \varphi\varphi'$. Then

$$\dot{\Delta}_j \Lambda^\sigma u = (\Lambda^\sigma \dot{\Delta}_j) \dot{\Delta}'_j u = 2^{j\sigma} \phi_j * (\dot{\Delta}'_j u),$$

where

$$\phi^\sigma := \mathcal{F}^{-1}(|\cdot|^\sigma \varphi) \quad \text{and} \quad \phi_j^\sigma(x) = 2^{dj} \phi^\sigma(2^j x).$$

For the low-frequency part, by the above identity,

$$\left| \sum_{j \leq k} \dot{\Delta}_j \Lambda^\sigma u(x) \right| \leq C \sum_{j \leq k} 2^{j\sigma} \|\dot{\Delta}'_j u\|_\infty \leq C \sum_{j \leq k} 2^{j(\sigma-\alpha)} \|u\|_{\dot{B}_{\infty, \infty}^\alpha}.$$

For the high-frequency part, we have

$$\begin{aligned} \left| \sum_{j>k} \dot{\Delta}_j \Lambda^\sigma u(x) \right| &\leq \sum_{j>k} \left| (\Lambda^{\sigma-s} \dot{\Delta}_j) \Lambda^s u(x) \right| = \sum_{j>k} 2^{-j(s-\sigma)} \left| \phi_j^{\sigma-s} * (\Lambda^s u)(x) \right| \\ &\leq C \left(\sum_{j>k} 2^{-j(s-\sigma)} \right) (\mathcal{M}\Lambda^s u)(x) \leq C 2^{-k(s-\sigma)} (\mathcal{M}\Lambda^s u)(x), \end{aligned} \quad (\text{B.4})$$

where we also used the fact that $|\phi_j^\alpha * f(x)| \leq C(\varphi, d, \alpha) \mathcal{M}f(x)$. Thus, we obtain

$$|\Lambda^\sigma u(x)| \leq C 2^{k(\sigma-\alpha)} \|u\|_{\dot{B}_{\infty,\infty}^\alpha} + C 2^{-k(s-\sigma)} (\mathcal{M}\Lambda^s u)(x). \quad (\text{B.5})$$

Choosing

$$k \approx (s-\alpha)^{-1} \log_2 \left(\frac{(\mathcal{M}\Lambda^s u)(x)}{\|u\|_{\dot{B}_{\infty,\infty}^\alpha}} \right)$$

and using (B.5), we obtain that

$$|\Lambda^\sigma u(x)| \leq [(\mathcal{M}\Lambda^s u)(x)]^\theta \|u\|_{\dot{B}_{\infty,\infty}^\alpha}^{1-\theta}, \quad \theta = \frac{\sigma-\alpha}{s-\alpha}.$$

Therefore, by (A.1) and (A.5), and noting that $\theta p = p_1 > 1$ and $\theta q = q_1 \geq 1$,

$$\begin{aligned} \|\Lambda^\sigma u\|_{L^{p,q}} &\leq \|\mathcal{M}\Lambda^s u\|_{L^{\theta p, \theta q}}^\theta \|u\|_{\dot{B}_{\infty,\infty}^\alpha}^{1-\theta} \\ &\leq \|\Lambda^s u\|_{L^{p_1, q_1}}^\theta \|u\|_{\dot{B}_{\infty,\infty}^\alpha}^{1-\theta} \quad \text{with } \theta = \frac{\sigma-\alpha}{s-\alpha} = \frac{p_1}{p} = \frac{q_1}{q}, \end{aligned}$$

which yields (B.2). \square

The following Ladyženskaja-type estimate for Lorentz space plays a crucial role in the proof of our main results when $d = 2$.

Lemma B.4 (Ladyženskaja-type estimate). *Let $d \geq 2$. For any $u \in W^{1,2}(\mathbb{R}^d)$, it holds that*

$$\|u\|_{L^{\frac{2d}{d-1}, 2}(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^d)}^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^d)}^{\frac{1}{2}}. \quad (\text{B.6})$$

Proof. Using (A.6), we derive

$$\|u\|_{L^{\frac{2d}{d-1}, 2}(\mathbb{R}^d)} \leq C \|\Lambda^{\frac{1}{2}} u\|_{L^2(\mathbb{R}^d)} \leq C \|\Lambda u\|_{L^2(\mathbb{R}^d)}^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^d)}^{\frac{1}{2}},$$

which yields (B.6). \square

The following lemma contains the refined version of Poincaré's estimates. Let

$$\underline{f}_A := \frac{1}{|A|} \int_A f \quad \text{and} \quad \#f\#_{L^p(A)} := \left(\int_A |f|^p \right)^{\frac{1}{p}}.$$

Lemma B.5 (Poincaré-type estimates). *Let $0 \leq \varphi \in C_c^\infty(B_1)$ such that $\int_{B_1} \varphi > 0$. Given $R > 0$, set $\varphi_R = \varphi(\cdot/R)$. Let*

$$\underline{u} = \left(\int_{\mathbb{R}^d} \varphi_R \right)^{-1} \int_{\mathbb{R}^d} u \varphi_R \quad \text{and} \quad \bar{u} = u - \underline{u}.$$

Then, the following holds:

(i) *For any $q \in [1, \infty)$,*

$$\#\bar{u}\#_{L^q(B_R)} \leq CR \#\nabla u\#_{L^q(B_R)}. \quad (\text{B.7})$$

(ii) For any $q \in [1, d)$ and $p \in [q, \frac{dq}{d-q}]$,

$$\|\bar{u}\|_{L^p(B_R)} \leq CR \|\nabla u\|_{L^q(B_R)}. \quad (\text{B.8})$$

Here the constant C only depends on d, p, q and φ .

Proof. By scaling, we may assume that $R = 1$. For any $q \in [1, \infty)$ and $p \in [q, \frac{dq}{d-q}]$, using Sobolev's estimate, we have

$$\|\bar{u}\|_{L^p(B_1)} \leq C \|\bar{u}\|_{W^{1,q}(B_1)} \leq C \|\bar{u}\|_{L^q(B_1)} + C \|\nabla \bar{u}\|_{L^q(B_1)}.$$

Thus, it suffices to prove

$$\|\bar{u}\|_{L^q(B_1)} \leq C \|\nabla u\|_{L^q(B_1)}, \quad q \in [1, \infty).$$

Arguing as in the proof of [14, Theorem 1 in Section 5.8.1], and assuming that the above estimate does not hold, we infer that there exists a sequence u_k such that

$$\underline{u}_k = 0, \quad \|u_k\|_{L^q(B_1)} = 1 \quad \text{and} \quad \|\nabla u_k\|_{L^q(B_1)} \leq \frac{1}{k}.$$

Thus, there exist a subsequence $\{u_{k_j}\}_{j \geq 1} \subseteq \{u_k\}_{k \geq 1}$ and $u \in L^q(B_1)$ such that $u_{k_j} \rightarrow u$ in $L^q(B_1)$ and $\underline{u} = 0$. Moreover, $\nabla u_{k_j} \rightharpoonup \nabla u$ in $L^q(B_1)$ and

$$\|\nabla u\|_{L^q(B_1)} \leq \liminf_{j \rightarrow \infty} \|\nabla u_{k_j}\|_{L^q(B_1)} = 0.$$

Therefore, u is a constant in B_1 . Taking into account $\underline{u} = \int_{B_1} u \varphi = 0$, we infer that $u = 0$, which however contradicts the fact that $\|u\|_{L^q(B_1)} = 1$. \square

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