An Axiomatization of Prospect Theory

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Abstract

Prospect theory is often used in theory and empirics, yet its axiomatic foundations are not fully understood. We offer a full axiomatic characterization of prospect theory. The key axiom, rank-independent trade-off consistency, is a rank-independent generalization of cumulative prospect theory's rank-dependent key axiom. In addition, prospect theory only requires within probability tree continuity and monotonicity for lotteries with the same probability tree capturing frame sensitivity and discontinuous changes across probability trees, whereas cumulative prospect theory assumes that the preferences are not affected by the probability tree.

1 Introduction

Prospect theory has been an immensely influential model for choice under risk. However, its axiomatic foundations are not well-understood. In this paper, we offer an axiomatization for prospect theory under risk. This provides a prospect theory counterpart for the axiomatization of cumulative prospect theory under risk in Chateauneuf and Wakker (1999).

Prospect theory (Kahneman and Tversky, 1979)[1] extends the expected utility by a probability weighting function. Formally, prospect theory uses a probability weighting function $w : [0,1] \to \mathbb{R}_+$ with $w(0) = 0$ and $w(1) = 1$ and a utility function u to evaluate lotteries P by

$$\sum_x w\Big(P(x)\Big)u(x)$$

where $P(x)$ is the probability of prize x in the lottery P. Here, the outcomes x are monetary gains or losses relative to a reference point. This model was originally introduced by Preston and Baratta (1948), Edwards (1954), and Handa (1977).

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[1]Following the extension to simple lotteries by Camerer and Ho (1994) and Fennema and Wakker (1996). This is also known as the separable prospect theory or prospect theory on positive and negative prizes (Wakker, 2010; Wakker, 2022).

We show that the main behavioral axiom for prospect theory is a stronger version of the main behavioral axiom for cumulative prospect theory in Chateauneuf and Wakker (1999). This axiom of trade-off consistency captures that the decision maker evaluates the value of trading off one of the prizes of the lottery for another prize consistently and independently of the probability of the prize and independently of the other prizes. In contrast, cumulative prospect theory assumes a restricted version of this axiom when the evaluation of the trade-off might depend on the rank of the prizes in the lottery.

As a second difference, we show that prospect theory only requires within probability tree continuity and monotonicity for lotteries with the same probability tree. This captures frame sensitivity and discontinuous changes in evaluations across probability trees. Instead, cumulative prospect theory assumes that the preferences are not affected by the probability tree of the lottery.

Related literature. Recently, there has been a discussion (Bernheim and Sprenger, 2020; 2023, Abdellaoui et al., 2020) on testing and comparing prospect theory and cumulative prospect theory. Our foundations for prospect theory clarify this discussion by formalizing the axiomatic difference between these two models. We show that the axiomatic difference between these models is rank-independent trade-off consistency of prospect theory compared to the first-order stochastic dominance and continuity across probability trees of cumulative prospect theory.

Cumulative prospect theory has several characterizations in Luce and Fishburn (1991), Tversky and Kahneman (1992), Wakker and Tversky (1993), Chateauneuf and Wakker (1999), Köbberling and Wakker (2003), Kothiyal et al. (2011), Schmidt and Zank (2012), Bastianello et al. (2023). However, the only characterization for prospect theory is a partial characterization in Kahneman and Tversky (1979) and an axiomatization for a linear utility in Handa (1977).[2] In here, we complete this characterization for a full characterization of prospect theory.

The theoretical literature on prospect theory has noted that it often violates first-order stochastic dominance (Fishburn, 1978). Hence, some authors have deemed it as unsuitable for theoretical analyses (Wakker, 2010, p. 275). However, prospect theory is often used in

[2]Handa's characterization crucially depends on linear utility, and it does not extend to non-linear utility.

empirical and theoretical applications, such as recently in Bernheim and Sprenger (2020), Blake et al. (2021), De Giorgi et al. (2007), Grishina et al. (2017), Harrison and Rutström (2009), Harrison et al. (2007), Hey et al. (2010), Schweitzer and Cachon (2000), Rieger (2014), Smith et al. (2009), and Wibbenmeyer et al. (2013). Our axiomatization of the prospect theory provides theoretical foundations for these empirical and theoretical applications.

Prospect theory is closely related to the subjectively weighted expected utility (Karmarkar, 1978; 1979). However, in contrast to this model, prospect theory does not normalize the weights to sum to 1.[3] This model does not have an axiomatization. Additionally, prospect theory is related to weighted expected utility (Chew and MacCrimmon, 1979; Chew, 1983; 1989; Hazen, 1987; Karni and Zhou, 2021). However, this model applies an additional weighting function to the outcomes and normalizes the weights to sum to 1.[4]

Organization of the paper. Section 2.1 introduces the notation and definitions. Section 2.2 introduces the axioms characterizing prospect theory and states the result of the paper. Section 2.3 compares the axioms of prospect theory to cumulative prospect theory. Section 2.4 sketches the proof. Section 3 concludes. The proof of the characterization is in the Appendix.

2 Characterization

2.1 Preliminaries

We consider a standard setting in choice under risk with monetary prizes without compound lotteries. The prizes are monetary prizes on an open interval around 0, $X = (m_*, m^*)$ where $m_*, m^* \in \mathbb{R} \cup \{\infty, -\infty\}$ and $m_* < 0 < m^*$. The set of (simple) lotteries on X is denoted by $\Delta(X)$. We consider preferences $\succsim$ over lotteries $P \in \Delta(X)$. $\operatorname{supp} P$ denotes the support of the lottery P.[5] We endow the set of lotteries $\Delta(X)$ with the topology of weak convergence.[6]

We define mixtures of lotteries prizewise which assumes that compound lotteries are

[3]Formally, the subjectively weighted expected utility uses utility u and weighting function for probabilities w such that the value of lottery P is $\sum_x \frac{w(P(x))u(x)}{\sum_y w(P(y))}$.

[4]Formally, the weighted expected utility uses utility u and weighting function for outcomes w such that the value of lottery P is $\sum_x \frac{P(x)w(x)u(x)}{\sum_y P(y)w(y)}$.

[5]For simple lotteries, $\operatorname{supp} P = \{x \in X | P(x) > 0\}$.

[6]For simple lotteries, weak convergence simplifies to the convergence of cumulative distribution functions outside the support of the limit lottery: A sequence of lotteries $(P_n)_{n=1}^{\infty} \subseteq \Delta(X)$ converges weakly to

reduced to single-stage lotteries: Define for all $\alpha \in [0,1]$, $P, Q \in \Delta(X)$, and $x \in X$,

$$\Big(\alpha P + (1-\alpha)Q\Big)(x) = \alpha P(x) + (1-\alpha)Q(x).$$

Our approach is to focus only on comparing lotteries with the same induced probability tree over outcomes. Formally:

Definition Lotteries P and Q *have the same induced probability tree* if there exist $p \in \Delta(\mathbb{N})$ and $(x_i)_{i\in \operatorname{supp} p}, (y_i)_{i\in \operatorname{supp} p} \in X^{\operatorname{supp} p}$ such that for all $i \neq j$, $x_i \neq x_j$ and $y_i \neq y_j$, $P = \sum_{i\in \operatorname{supp} p} p_i \delta_{x_i}$, and $Q = \sum_{i\in \operatorname{supp} p} p_i \delta_{y_i}$.

In this definition, the probability trees of the lotteries are only compared based on the observable probabilities for each outcome. That is, after the reduction of compound lotteries and coalescing common prizes.

2.2 Prospect Theory

We characterize prospect theory by the following six axioms. Our first four assumptions are standard or technical continuity assumptions. The first two assumptions are that preferences are complete, transitive, and monotone in increasing prizes within the induced probability tree. This weaker notion of monotonicity is the first difference to cumulative prospect theory that satisfies monotonicity in first-order stochastic dominance.

Axiom 1 (Weak Order) $\succsim$ is complete and transitive.

Axiom 2 (Single Prize Monotonicity) For all lotteries P, $x, y \notin \operatorname{supp} P$ with $x > y$ and $\alpha \in (0,1)$,

$$\alpha P + (1-\alpha)\delta_x \succ \alpha P + (1-\alpha)\delta_y.$$

The next two axioms are technical continuity axioms in prizes and in probabilities. First, we assume continuity in prizes when the probabilities of the prizes are unchanged. That is when the induced probability trees of the lotteries are the same.[7]

$P \in \Delta(X)$ if for all $a \in \mathbb{R}$ with $a \notin \operatorname{supp} P$,

$$\sum_{\substack{x\in \operatorname{supp} P_n \\ x \le a}} P_n(x) \to \sum_{\substack{x\in \operatorname{supp} P \\ x \le a}} P(x) \text{ as } n \to \infty.$$

[7]Convergence for lotteries with the same induced probability tree is the same as prizewise convergence: If lotteries $(P_n)_{n=1}^{\infty}$, and P have the same induced probability tree $p \in \Delta(\mathbb{N})$, then P_n converges weakly to P as $n \to \infty$ iff for each $n \in \mathbb{N}$, there exist $(x_i^n)_{i\in \operatorname{supp} p}, (x_i)_{i\in \operatorname{supp} p} \in X^{\operatorname{supp} p}$ such that $P^n = \sum_{i\in \operatorname{supp} p} p_i \delta_{x_i^n}$, $P = \sum_{i\in \operatorname{supp} p} p_i \delta_{x_i}$, and for all $i \in \operatorname{supp} p$, $x_i^n \to x_i$ as $n \to \infty$ in the Euclidean distance.

Axiom 3 (Within Tree Continuity in Prizes) For all lotteries $(P_n)_{n=1}^{\infty}$, P, and Q such that for all $n \in \mathbb{N}$, P_n and P have the same induced probability tree and P_n converges weakly to P as $n \to \infty$,

$$\text{if } P_n \succsim Q \text{ for all } n \in \mathbb{N}, \text{ then } P \succsim Q \text{ and if } Q \succsim P_n \text{ for all } n \in \mathbb{N}, \text{ then } Q \succsim P.$$

The next axiom is continuity in the probabilities when the prizes are unchanged. This is the standard mixture continuity.

Axiom 4 (Mixture Continuity) For all lotteries P, Q, R, the sets

$$\left\{\alpha \in [0,1] \middle| \alpha P + (1-\alpha)Q \succsim R\right\} \text{ and } \left\{\alpha \in [0,1] \middle| R \succsim \alpha P + (1-\alpha)Q\right\} \text{ are closed.}$$

The next axiom is the main axiom of the representation. It is a stronger version of the trade-off consistency axiom from Chateauneuf and Wakker (1999) when we apply the axiom to all the lotteries and not only lotteries that have the same induced probability tree and rank prizes of the lotteries in the same order. The interpretation of this axiom is that the decision-maker ranks utility differences between prizes consistently and independently of the other prizes or of the probabilities. In contrast, the weaker version from Chateauneuf and Wakker (1999) allows the rank of a utility difference to depend on the rank of the prize.[8]

Axiom 5 (Rank-Independent Trade-Off Consistency) For all lotteries P, Q, R, S, $\alpha, \beta \in (0,1)$, $a, c \notin \operatorname{supp} P \cup \operatorname{supp} R$, and $b, d \notin \operatorname{supp} Q \cup \operatorname{supp} S$, if

$$\alpha\delta_a + (1-\alpha)P \succsim \alpha\delta_b + (1-\alpha)Q, \quad \alpha\delta_d + (1-\alpha)Q \succsim \alpha\delta_c + (1-\alpha)P,$$
$$\text{and } \beta\delta_c + (1-\beta)R \succsim \beta\delta_d + (1-\beta)S,$$

then $\beta\delta_a + (1-\beta)R \succsim \beta\delta_b + (1-\beta)S$.

The idea of the axiom is to capture utility differences between the prizes. Assume that we have additively separable utility across the prizes. From the first two preferences, we can infer that trading b for a compensates for trading Q for P. Whereas trading d for c does not compensate for trading Q for P. Hence, the utility difference between a and b is larger than the utility difference between c and d. So as in the third preference, if trading d for c compensates for trading S for R, then we must have that also trading b for a compensates

[8] This axiom was initially introduced in Wakker (1984) in choice under uncertainty for subjective expected utility.

for trading S for R, as is assumed in the last preference.

The last axiom captures that the outcomes are gains and losses with respect to a reference level 0.

Axiom 6 (Gain-Loss Monotonicity) For all lotteries $P \in \Delta(X)$, if for all $x \in \operatorname{supp} P$, $x > 0$, then $P \succ \delta_0$ and if for all $x \in \operatorname{supp} P$, $x < 0$, then $\delta_0 \succ P$.

Our main result is that these six axioms characterize prospect theory.

Theorem 1 (Prospect Theory) $\succsim$ satisfies Axioms 1-6 iff there exist a continuous and strictly increasing $u : X \to \mathbb{R}$ with $u(0) = 0$ and a continuous probability weighting function $w : [0,1] \to \mathbb{R}_+$ with $w(0) = 0$ and $w(1) = 1$ such that

$$P \succsim Q \iff \sum_{x \in \operatorname{supp} P} w\Big(P(x)\Big)u(x) \geq \sum_{x \in \operatorname{supp} Q} w\Big(Q(x)\Big)u(x).$$

This characterization clarifies the axiomatic foundations of prospect theory. We discuss this result more in the next section and the role of Axioms 5 and 6 in our proof sketch in Section 2.4.

2.3 Comparison to Cumulative Prospect Theory

The difference between prospect theory (PT) and cumulative prospect theory (CPT) (Chateauneuf and Wakker, 1999) is that CPT trades off monotonicity and continuity for rank-dependent evaluation of prizes in trade-off consistency. First, CPT strengthens Axiom 2 to apply for all prizes x, y and not only prizes that are not a part of the lottery P, as is assumed in PT. Second, CPT strengthens Axiom 3 to apply for all limit lotteries P and not only limit lotteries P that have the same induced probability tree as each P_n, as PT assumes. These two differences highlight the view in PT that the preferences are affected by the induced probability tree. PT views the induced probability tree as a frame for evaluating the lotteries, and lotteries are evaluated consistently, monotonically and continuously, within the same frame. However, continuous changes in the prizes that change the induced probability can create discontinuous changes in preferences. In contrast, CPT assumes that the induced probability tree does not affect the preferences.

The third difference is that CPT assumes a weaker version of Axiom 5. CPT weakens this axiom in three ways. First, it is assumed that P and Q have the same induced probability tree and they rank the branches in the same order. Formally, there exist $p \in \Delta(\mathbb{N})$ $(x_i)_{i \in \operatorname{supp} p}$,

$(y_i)_{i\in\operatorname{supp} p} \in X^{\operatorname{supp} p}$ such that if $i < j$, then $x_i > x_j$ and $y_i > y_j$, and $P = \sum_{i\in\operatorname{supp} p} p_i\delta_{x_i}$, and $Q = \sum_{i\in\operatorname{supp} p} p_i\delta_{y_i}$. Second, it is assumed that a, b, c, d have the same rank as outcomes of P and Q: Formally, for each $\theta \in \{a, b, c, d\}$, for all $i \in \operatorname{supp} p$, $\theta \geq x_i$ iff $\theta \geq y_i$. Third, it is assumed that a, b, c, d are all negative or are all positive. Finally, the first and second properties are also assumed to hold for R, S and a, b, c, d.

First, PT allows for comparisons of P and Q that have different induced probability trees.[9] This assumes that if two lotteries share a common prize with a common probability, then changing this common prize to another common prize does not affect the comparison between them.[10] Second, PT allows for comparisons of prizes with different ranks for a rank-independent weighting function. This is the well-known difference between the models and this assumption was tested recently by Bernheim and Sprenger (2020). Third, PT allows for comparisons of prizes that are losses and gains for a sign-independent weighting function. It is a simple extension of PT to extend it for sign-dependent weighting function by applying the trade-off consistency separately for gains and losses, as in Chateauneuf and Wakker (1999).

2.4 Proof Sketch

The proof follows in three steps. The first part of the proof offers a representation for more general preferences by relaxing Axioms 5 and 6 by the branch cancellation axiom: For two lotteries P, Q, if $x, y \notin \operatorname{supp} P \cup \operatorname{supp} Q$ and $\alpha \in (0, 1)$, then

$$\alpha P + (1-\alpha)\delta_x \succsim \alpha Q + (1-\alpha)\delta_x \iff \alpha P + (1-\alpha)\delta_y \succsim \alpha Q + (1-\alpha)\delta_y.$$

We show these preferences are represented by $u : [0, 1] \times X \to \mathbb{R}$ with $u(p, 0) = 0$ for all p and $c : \Delta(X) \to \mathbb{R}$ that only depends on the induced probability tree such that the value of lottery P is $\sum_{x\in\operatorname{supp} P} u(P(x), x) + C(P)$. Here, the branch cancellation axiom is similar to the separability axiom for additive representations (Debreu, 1960) and to the weak c-independence axiom in Maccheroni et al. (2006) that gives a similar variational component

[9] This assumption is crucial since it assumes that utility changes are evaluated similarly across different induced probability trees. Otherwise, our axioms are compatible with a model where the value of lottery P is $\alpha(P)[\sum_{x\in\operatorname{supp} P} w\big(P(x)\big)u(x)]$ where the function α depends only on the induced probability tree of P.

[10] Formally, for two lotteries P, Q, if $x, y \notin \operatorname{supp} P \cup \operatorname{supp} Q$ and $\alpha \in (0, 1)$, then

$$\alpha P + (1-\alpha)\delta_x \succsim \alpha Q + (1-\alpha)\delta_x \iff \alpha P + (1-\alpha)\delta_y \succsim \alpha Q + (1-\alpha)\delta_y.$$

to the representation.

The second step strengthens the branch cancellation axiom with Axiom 5. Then this corresponds to separating the probability and utility of outcomes for a representation by $u : X \to \mathbb{R}$ with $u(0) = 0$, $w : [0, 1] \to \mathbb{R}_+$, and $c : \Delta(X) \to \mathbb{R}$ that only depends on the induced probability tree such that the value of lottery P is $\sum_{x \in \operatorname{supp} P} w(P(x))u(x) + C(P)$. This follows symmetrically to Wakker (1984).

The third step adds Axiom 6 and shows that this gets rid of the additive term since it is not compatible with the normalization of $u(0) = 0$ and continuity of prizes around 0. This gives us the final prospect theory representation.

3 Conclusion

In this paper, we axiomatized prospect theory (PT) under risk. This result provided a counterpart for Chateauneuf and Wakker (1999). We showed that the main behavioral axiom of PT is a stronger version of the main behavioral axiom of cumulative prospect theory (CPT). It is left for future research to axiomatize PT under uncertainty to provide a counterpart for Wakker and Tversky (1993). It is unclear if the different axiomatizations of CPT, such as in Wakker and Tversky (1993), Köbberling and Wakker (2003), Kothiyal et al. (2011), Schmidt and Zank (2012), and Bastianello et al. (2023) can be extended for an axiomatization of PT similarly as in here. Additionally, our approach can be extended for axiomatizations of generalizations of CPT and PT, such as CPT-simplicity model and PT-simplicity model (Fudenberg and Puri, 2022).

Appendix to "An Axiomatization of Prospect Theory"

This appendix shows the proof for Theorem 1. First, Section A.1 starts with definitions and notations used in the proof. Sections A.2-A.4 show the first step of the proof sketch. Section A.2 shows that lotteries within an induced probability tree have an additive representation with an inseparable utilites and probabilities. Section A.3 shows that we can compare utilities across probability trees with the additive representation locally in a preference interval. Section A.4 shows that we can extend this local representation into a global one after adding an additive induced probability tree dependent constant. Finally, Section A.5

shows the second and third steps of the proof sketch.

A Proofs

A.1 Definitions and notation

We use the following notation. For an induced probability tree, $p \in \Delta(\mathbb{N})$, denote the off-diagonal of $X^{\operatorname{supp} p}$ by

$$\text{NDiag}(X^p) = \{x \in X^{\operatorname{supp} p} | \, \forall \, i, j \in \operatorname{supp} p, i \neq j, x_i \neq x_j\}.$$

For $p \in \Delta^4(\mathbb{N})$, $i, j \in \operatorname{supp} p$, $x \in \text{NDiag}(X^p)$, $c \in X$, we denote replacing the value of x by c in the coordinate i by (c_i, x_{-i}). In a vector, the outermost coordinate denotes the location of the value in the vector. For example, switching the values of coordinates i and j in x is denoted by $((x_j)_i, (x_j)_i, x_{-i,j})$.

We denote the lottery with an induced probability tree p and prizes $x \in X^{\operatorname{supp} p}$ by

$$(p, x) = \sum_{i \in \operatorname{supp} p} p_i \delta_{x_i}.$$

We focus on preferences conditional on an induced probability tree p, $\succsim^p$. These are preferences on $\text{NDiag}(X^p)$ and are defined by for all $x, y \in \text{NDiag}(X^p)$,

$$x \succsim^p y \iff (p, x) \succsim (p, y).$$

We denote induced probability trees with at least 4 branches by

$$\Delta^4(\mathbb{N}) = \{p \in \Delta(\mathbb{N}) || \operatorname{supp} p| \geq 4\}.$$

For each $p \in \Delta^4(\mathbb{N})$, we say that $(u_i)_{i \in \operatorname{supp} p}$ is an *additive representation for* $\succsim^p$ if for each $i \in \operatorname{supp} p$, $u_i : X \to \mathbb{R}$ is a strictly increasing and continuous function such that for all $x, y \in \text{NDiag}(X^p)$,

$$x \succsim^p y \iff \sum_{i \in \operatorname{supp} p} u_i(x_i) \geq \sum_{i \in \operatorname{supp} p} u_i(y_i).$$

For the first part of the proof, we use the following weaker axiom of branch cancellation instead of tradeoff consistency.

Axiom 7 (Branch Cancellation) For two lotteries P, Q, if $x, y \notin \operatorname{supp} P \cup \operatorname{supp} Q$ and $\alpha \in (0,1)$, then

$$\alpha P + (1-\alpha)\delta_x \succsim \alpha Q + (1-\alpha)\delta_x \iff \alpha P + (1-\alpha)\delta_y \succsim \alpha Q + (1-\alpha)\delta_y.$$

A.2 Same Utility Across Induced Probability Trees

By Mononen (2024), we have the following result for an additive representation within each induced probability tree.

Proposition 2 Assume that $\succsim$ satisfies Axioms 1-4 and 7. Then for each $p \in \Delta^4(\mathbb{N})$, there exists $(u^p(p_i, \cdot))_{i \in \operatorname{supp} p}$ that is an additive representation for $\succsim^p$.

In the next two lemmas, we show that for all $p, q \in \Delta^4(\mathbb{N})$, if $p_i = q_j$, then $u^p(p_i, \cdot) = u^p(q_i, \cdot)$.

Lemma 3 Assume that $\succsim$ satisfies Axioms 1-4 and 7 and for each $p \in \Delta^4(\mathbb{N})$, $\succsim^p$ has an additive representation by $(u^p(p_i, \cdot))_{i \in \operatorname{supp} p}$. If $p, q \in \Delta^4(\mathbb{N})$ and $i, j \in \operatorname{supp} p \cap \operatorname{supp} q$ are such that $p_i = q_i$ and $p_j = q_j$, then there exist $\eta > 0$, $\beta_1, \beta_2 \in \mathbb{R}$ such that $\eta u^p(p_i, \cdot) + \beta_1 = u^q(q_i, \cdot)$ and $\eta u^p(p_j, \cdot) + \beta_2 = u^q(q_j, \cdot)$.

Proof. For all $\alpha \in [0,1]$, we denote $q\alpha p = \alpha q + (1-\alpha)p$. Denote
$A = \Big\{\alpha \in [0,1] \Big| \exists\, \eta > 0, \beta_1, \beta_2 \in \mathbb{R}, \eta u^p(p_i, \cdot) + \beta_1 = u^{q\alpha p}(p_i, \cdot), \eta u^p(p_j, \cdot) + \beta_2 = u^{q\alpha p}(p_j, \cdot)\Big\}$.

We show that $A = [0,1]$. $0 \in A$. First, we show that if $\alpha_0 \in A \cap [0,1)$, then there exist $\varepsilon > 0$ such that for all $\alpha_1 \in (\alpha_0, \alpha_0 + \varepsilon)$, $\alpha_1 \in A$. Let $k \in \operatorname{supp}(q\alpha_0 p) \setminus \{i, j\}$. Let $x \in \operatorname{NDiag}(X^{p \cup q})$.[11] Let $x_k^1, x_k^2 \in X$ be such that $x_k^1 < x_k < x_k^2$ and for all $\tilde{x}_k \in [x_k^1, x_k^2]$, $(\tilde{x}_k, x_{-k}) \in \operatorname{NDiag}(X^{p \cup q})$. By Axiom 2, $\big(q\alpha_0 p, (x_k^2, x_{-k})\big) \succ (q\alpha_0 p, x) \succ \big(q\alpha_0 p, (x_k^1, x_{-k})\big)$. By Axiom 4, there exist $\varepsilon > 0$ such that for all $\alpha_1 \in (\alpha_0, \alpha_0 + \varepsilon)$, $\big(q\alpha_0 p, (x_k^2, x_{-k})\big) \succ (q\alpha_1 p, x) \succ \big(q\alpha_0 p, (x_k^1, x_{-k})\big)$. Let $\alpha_1 \in (\alpha_0, \alpha_0 + \varepsilon)$. By Axiom 3, there exist $\tilde{x}_k \in [x_k^1, x_k^2]$ such that $\big(q\alpha_0 p, (\tilde{x}_k, x_{-k})\big) \sim (q\alpha_1 p, x)$.

Denote $Y = X \setminus (\{x_k | k \in \operatorname{supp} p \cup q\} \cup \{\tilde{x}_k\})$ and $Z = \{(x, y) \in Y \times Y | x \neq y\}$. Now $(\operatorname{cl} Z) \cap X = X$. Let $(c_i, d_j), (\tilde{c}_i, \tilde{d}_j) \in Z$. By Axiom 7, we have
$\big(q\alpha_0 p, (c_i, d_j, \tilde{x}_k, x_{-i,j,k})\big) \sim \big(q\alpha_1 p, (c_i, d_j, x_{-i,j})\big), \big(q\alpha_0 p, (\tilde{c}_i, \tilde{d}_j, \tilde{x}_k, x_{-i,j,k})\big) \sim \big(q\alpha_1 p, (\tilde{c}_i, \tilde{d}_j, x_{-i,j})\big)$.

[11] This denotes $\{x \in X^{\operatorname{supp} p \cup \operatorname{supp} q} | \forall i, j \in \operatorname{supp} p \cup \operatorname{supp} q, i \neq j, x_i \neq x_j\}$

Thus by the additive representation, we have

$$u^{q\alpha_0 p}(p_i, c_i) + u^{q\alpha_0 p}(p_j, d_j) \geq u^{q\alpha_0 p}(p_i, \tilde{c}_i) + u^{q\alpha_0 p}(p_j, \tilde{d}_j)$$
$$\iff u^{q\alpha_1 p}(p_i, c_i) + u^{q\alpha_1 p}(p_j, d_j) \geq u^{q\alpha_1 p}(p_i, \tilde{c}_i) + u^{q\alpha_1 p}(p_j, \tilde{d}_j).$$

By the continuity of $u^{q\alpha_0 p}$ and $u^{q\alpha_1 p}$ and $(\operatorname{cl} Z) \cap X = X$, we have for all $(c_i, d_j), (\tilde{c}_i, \tilde{d}_j) \in X^2$

$$u^{q\alpha_0 p}(p_i, c_i) + u^{q\alpha_0 p}(p_j, d_j) \geq u^{q\alpha_0 p}(p_i, \tilde{c}_i) + u^{q\alpha_0 p}(p_j, \tilde{d}_j)$$
$$\iff u^{q\alpha_1 p}(p_i, c_i) + u^{q\alpha_1 p}(p_j, d_j) \geq u^{q\alpha_1 p}(p_i, \tilde{c}_i) + u^{q\alpha_1 p}(p_j, \tilde{d}_j).$$

Define the order $\trianglerighteq$ on X^2 by for all $(c_i, d_j), (\tilde{c}_i, \tilde{d}_j) \in X^2$

$$(c_i, d_j) \trianglerighteq (\tilde{c}_i, \tilde{d}_j) \iff u^{q\alpha_0 p}(p_i, c_i) + u^{q\alpha_0 p}(p_j, d_j) \geq u^{q\alpha_0 p}(p_i, \tilde{c}_i) + u^{q\alpha_0 p}(p_j, \tilde{d}_j).$$

Now $\Big(u^{q\alpha_0 p}(p_i, \cdot), u^{q\alpha_0 p}(p_j, \cdot)\Big)$ and $\Big(u^{q\alpha_1 p}(p_i, \cdot), u^{q\alpha_1 p}(p_j, \cdot)\Big)$ are additive continuous representations for $\trianglerighteq$. Thus by the uniqueness of an additive representation (Krantz et al., 1971), there exist $\eta > 0$, $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$\eta u^{q\alpha_0 p}(p_i, \cdot) + \beta_1 = u^{q\alpha_1 p}(p_i, \cdot) \text{ and } \eta u^{q\alpha_0 p}(p_i, \cdot) + \beta_2 = u^{q\alpha_1 p}(p_i, \cdot).$$

Next, assume that there exists $(\alpha^l)_{l \in \mathbb{N}} \subseteq A$ such that $\alpha^j \to \alpha_0$. We show that $\alpha_0 \in A$. Assume w.l.o.g. that the sequence $(\alpha^l)_{l \in \mathbb{N}}$ is decreasing. Let $k \in \operatorname{supp}(q\alpha_0 p) \setminus \{i, j\}$. Let $x \in \operatorname{NDiag}(X^{p \cup q})$. Let $x_k^1, x_k^2 \in X$ be such that $x_k^1 < x_k < x_k^2$ for all $\tilde{x}_k \in [x_k^1, x_k^2]$, $(\tilde{x}_k, x_{-k}) \in \operatorname{NDiag}(X^{p \cup q})$. By Axiom 2, $\Big(q\alpha_0 p, (x_k^2, x_{-k})\Big) \succ (q\alpha_0 p, x) \succ \Big(q\alpha_0 p, (x_k^1, x_{-k})\Big)$. By Axiom 4, there exist $\varepsilon > 0$ such that for all $\alpha_1 \in (\alpha_0, \alpha_0 + \varepsilon)$, $\Big(q\alpha_0 p, (x_k^2, x_{-k})\Big) \succ (q\alpha_1 p, x) \succ \Big(q\alpha_0 p, (x_k^1, x_{-k})\Big)$. Let $l \in \mathbb{N}$ be such that $\alpha^l \in (\alpha_0, \alpha_0 + \varepsilon)$. By Axiom 3, there exist $\tilde{x}_k \in [x_k^1, x_k^2]$ such that $\Big(\alpha_0 p, (\tilde{x}_k, x_{-k})\Big) \sim (q\alpha^l p, x)$. Now the claim $\alpha_0 \in A$ follows similarly as above. □

Next, we show that we can normalize the utility functions in the additive representations so that the utility functions for the same probabilities are equal across the induced probability trees.

Proposition 4 Assume that $\succsim$ satisfies Axioms 1-4 and 7. Then there exists $u\colon (0,1) \times X \to \mathbb{R}$ such that for all $p \in \Delta^4(\mathbb{N})$, $(u(p_i, \cdot))_{i \in \operatorname{supp} p}$ is an additive representation for $\succsim^p$.

Proof. By Proposition 2, for each $p \in \Delta^4(\mathbb{N})$, there exist an additive representation $(u^p(p_i, \cdot))$ for $\succsim^p$.

By the uniqueness of additive representations (Krantz et al., 1971), we can normalize these additive representations as follows. Let $x^* \in X$ be such that $0 < x^*$. First, for each $p \in \Delta^4(\mathbb{N})$ and $i \in \operatorname{supp} p$, normalize $u^p(p_i, \cdot)$ by $u^p(p_i, 0) = 0$ by an additive transformation. Second, for each $p \in \Delta^4(\mathbb{N})$ if there exist $i \in \operatorname{supp} p$ such that $p_i = 1/4$, normalize $u^p(1/4, \cdot)$ by $u^p(p_i, x^*) = 1$ by a common scaling.

For $a < 1/2$, let $\theta(a) \in \Delta^4(\mathbb{N})$ be such that $\theta(a)_1 = a$ and $\theta(a)_2 = 1/4$. For $a \geq 1/2$, let $\theta(a) \in \Delta^4(\mathbb{N})$ be such that $\theta(a)_1 = a$. Let $p \in \Delta^4(\mathbb{N})$ be such that for all $i \in \operatorname{supp} p$, $p_i \neq \frac{1}{4}$. Denote $i^* = \arg\min_{i \in \operatorname{supp} p} p_i$. Now especially $p_{i^*} < \frac{1}{2}$. Normalize u^p by a scaling such that $u^p(p_{i^*}, x^*) = u^{\theta(a)}(a, x^*)$. For each $a \in (0,1)$, denote $u(a, \cdot) = u^{\theta(a)}(a, \cdot)$. We show that for all $p \in \Delta^4(\mathbb{N})$ and $i \in \operatorname{supp} p$, $u^p(p_i, \cdot) = u(p_i, \cdot)$.

First, assume that there exists $j \in \operatorname{supp} p$ with $j \neq i$ and $p_j = \frac{1}{4}$. Then by Lemma 3 and the normalization, $u^p(p_i, \cdot) = u^{\theta(p_i)}(p_i, \cdot)$ and $u^p(\frac{1}{4}, \cdot) = u^{\theta(p_i)}(\frac{1}{4}, \cdot)$.

Next, assume that $p_i < \frac{1}{2}$. Let $j \in \arg\min\{p_l | l \in \operatorname{supp} p \setminus \{i\}\}$ and $q \in \Delta^4(\mathbb{N})$ be such that $q_i = p_i$, $q_j = p_j$ and there exists $k \in \operatorname{supp} q$ with $k \neq i$, $k \neq j$, and $q_k = \frac{1}{4}$. q exists since $p_j \leq \frac{1-p_i}{3}$, $p_i < \frac{1}{2}$. By the first case, $u^q(q_i, \cdot) = u(q_i, \cdot)$ and $u^q(q_j, \cdot) = u(q_j, \cdot)$. Now by the normalization, $u^p(p_i, 0) = u(p_i, 0)$, $u^p(p_j, 0) = u(p_j, 0)$, and if $i \in \arg\min\{p_l | l \in \operatorname{supp} p\}$, $u^p(p_i, x^*) = u(p_i, x^*)$ and otherwise $u^p(p_j, x^*) = u(p_j, x^*)$. Thus by Lemma 3 for p and q, we have $u^p(p_i, \cdot) = u(p_i, \cdot)$ and $u^p(p_j, \cdot) = u(p_j, \cdot)$.

Finally, assume that $p_i \geq \frac{1}{2}$. Let $j \in \arg\min\{p_l | l \in \operatorname{supp} p\}$ and $c = \min\{\theta(p_i)_l | l \in \operatorname{supp} \theta(p_i)\}$. Let $q \in \Delta^4(\mathbb{N})$ be such that $q_1 = p_i$, $q_2 = p_j$, and $q_3 = c$. By the previous case, $u^q(p_j, \cdot) = u(p_j, \cdot)$ and $u^q(c, \cdot) = u(c, \cdot)$. By the normalizations, we have $u^p(p_j, x^*) = u(p_j, x^*)$, $u^p(p_i, 0) = u(p_i, 0) = u^q(p_i, 0)$, and $u^p(p_j, 0) = u(p_j, 0)$. Thus by Lemma 3, $u^p(p_i, \cdot) = u^q(p_i, \cdot)$ and $u^p(p_j, \cdot) = u^q(p_j, \cdot)$. Symmetrically, by the normalization and Lemma 3, $u^{\theta(p_i)}(p_i, \cdot) = u^q(p_i, \cdot)$ and $u^{\theta(p_i)}(p_j, \cdot) = u^q(p_j, \cdot)$. Thus $u^p(p_i, \cdot) = u^{\theta(p_i)}(p_i, \cdot)$. This shows the claim. □

A.3 Local Additive Representation

In this section, we show that the preferences have locally additive representation. This means that for a lotteries $(p, x) \sim (q, y)$, there exist a preference interval $P \succ (p, x) \succ Q$ such that for all $\tilde{x} \in \operatorname{NDiag}(X^p), \tilde{y} \in \operatorname{NDiag}(X^q)$,

$$(p, \tilde{x}) \succsim (q, \tilde{y}) \iff \sum_{i \in \operatorname{supp} p} u(p_i, \tilde{x}_i) - \sum_{i \in \operatorname{supp} p} u(p_i, x_i) \geq \sum_{i \in \operatorname{supp} q} u(q_i, \tilde{y}_i) - \sum_{i \in \operatorname{supp} q} u(q_i, y_i).$$

This result follows in two steps. The first part shows locally additive representation for two induced probability trees that share a common probability and a common preference range where the two induced probability trees share a common prize with a common probability at the boundaries of the preference range. The idea of the proof is that we can use this common probability and the common prizes to measure the length of the preference interval using Axiom 7 in terms of both of the induced probability trees. This proof idea has been illustrated e.g. in Wakker (1989).

Lemma 5 Assume that $\succsim$ satisfies Axioms 1-4 and 7 and there exists $u:(0,1)\times X\to\mathbb{R}$ such that for all $p\in\Delta^4(\mathbb{N})$, $(u(p_i,\cdot))_{i\in\operatorname{supp}p}$ is an additive representation for $\succsim^p$.

If $p,q\in\Delta^4(\mathbb{N}), i^*\in\operatorname{supp}p, j^*\in\operatorname{supp}q$, are such that $p_{i^*}=q_{j^*}$ and there exist $x^1,x^2,x^3\in\operatorname{NDiag}(X^p)$ and $y^1,y^2,y^3\in\operatorname{NDiag}(X^q)$ such that $(p,x^1)\sim(q,y^1)$, $(p,x^2)\sim(q,y^2)$, $(p,x^3)\sim(q,y^3)$, $(p,x^1)\succsim(p,x^2)\succsim(p,x^3)$, $x^1_{i^*}=y^1_{j^*}$, and $x^3_{i^*}=y^3_{j^*}$, then for all $x\in\operatorname{NDiag}(X^p), y\in\operatorname{NDiag}(X^q)$ such that $(p,x^1)\succsim(p,x),(q,y)\succsim(p,x^3)$,

$$(p,x)\succsim(q,y)\iff\sum_{i\in\operatorname{supp}p}u(p_i,x_i)-\sum_{i\in\operatorname{supp}p}u(p_i,x^2_i)\geq\sum_{i\in\operatorname{supp}q}u(q_i,y_i)-\sum_{i\in\operatorname{supp}q}u(q_i,y^2_i).$$

Proof. By permuting the indices for p and q, we can assume that $i^*=j^*=1$. First, we show that we can assume that $x^1_1>x^3_1$: Since X is open there exist $\tilde{x}^1_1,\tilde{x}^3_1\in X$ such that $\tilde{x}^1_1>\max\{x^1_1,x^3_1\}\geq\min\{x^1_1,x^3_1\}>\tilde{x}^3_1$ and by Axiom 7,

$$\Big(p,(\tilde{x}^1_i,x^1_{-i})\Big)\sim\Big(p,(\tilde{x}^1_i,y^1_{-i})\Big)\succsim(p,x^1)\succsim(p,x^3)\succsim\Big(p,(\tilde{x}^3_i,x^3_{-i})\Big)\sim\Big(p,(\tilde{x}^3_i,y^3_{-i})\Big).$$

So now we could prove the claim for this larger interval instead. So assume that $x^1_1>x^3_1$.

Assume w.l.o.g. that i as above is 1. Since u is continuous and $|\operatorname{supp}p|\geq 4$, there exist $\tilde{x}^1\in\operatorname{NDiag}(X^p)$, $\tilde{y}^1\in\operatorname{NDiag}(X^q)$ such that $\tilde{x}^1_1=\tilde{y}^1_1=x^1_1$, for all $i\in\operatorname{supp}p, j\in\operatorname{supp}q$, $\tilde{x}^1_i\neq x^3_1\neq\tilde{y}^1_j$,

$$\sum_{i\in\operatorname{supp}p}u\Big(p_i,\tilde{x}^1_i\Big)=\sum_{i\in\operatorname{supp}p}u(p_i,x^1_i),\text{ and }\sum_{i\in\operatorname{supp}q}u\Big(q_i,\tilde{y}^1_i\Big)=\sum_{i\in\operatorname{supp}q}u(q_i,y^1_i).\tag{1}$$

Now by the additive representations for $\succsim^p$ and $\succsim^q$,

$$(p,\tilde{x}^1)\sim(p,x^1)\sim(q,y^1)\sim(q,\tilde{y}^1).\tag{2}$$

Let $n^0\in\mathbb{N}$ be the smallest positive integer such that

$$n^0\geq\frac{\sum_{i\in\operatorname{supp}p\setminus\{1\}}u(p_i,x^1_i)-\sum_{i\in\operatorname{supp}p\setminus\{1\}}u(p_i,x^3_i)}{\Big(u(p_1,x^1_1)-u(p_1,x^3_1)\Big)}.$$

We define by induction for each $1\leq n\leq n^0$, $\varphi(p,n)\in\operatorname{NDiag}(X^p),\varphi(q,n)\in\operatorname{NDiag}(X^q)$ such that $\varphi(p,n)_1=\varphi(q,n)_1=x^3_1$, for all $i\in\operatorname{supp}p, j\in\operatorname{supp}q$, $\varphi(p,n)_i\neq x^1_1$, $\varphi(q,n)_j\neq x^1_1$,

$\Big(p,\varphi(p,n)\Big)\sim\Big(q,\varphi(q,n)\Big)$, and for each $1\leq n\leq n^0$, $\Big(p,(x_1^1,\varphi(p,n)_{-1})\Big)\sim\Big(p,\varphi(p,n+1)\Big)$. By Axiom 7, we especially have then for each $1\leq n\leq n^0$, $\Big(q,(x_1^1,\varphi(q,n)_{-1})\Big)\sim\Big(q,\varphi(q,n+1)\Big)$.

First, since u is continuous and $|\operatorname{supp} p|\geq 4$, there exists $\varphi(p,1)\in \mathrm{NDiag}(X^p)$, $\varphi(q,1)\in \mathrm{NDiag}(X^q)$ such that $\varphi(q,1)_1=\varphi(p,1)_1=x_1^3$, for all $i\in\operatorname{supp} p$, $j\in\operatorname{supp} q$, $\varphi(p,1)_i\neq x_1^1\neq \varphi(q,1)_j$,

$$\sum_{i\in\operatorname{supp} p} u\Big(p_i,\varphi(p,1)_i\Big)=\sum_{i\in\operatorname{supp} p} u(p_i,x_i^3),\ \text{and} \sum_{i\in\operatorname{supp} q} u\Big(q_i,\varphi(q,1)_i\Big)=\sum_{i\in\operatorname{supp} q} u(q_i,y_i^3). \tag{3}$$

By the additive representations for $\succsim^p$ and $\succsim^q$, $\Big(p,\varphi(p,1)\Big)\sim(p,x^3)\sim(q,y^3)\sim\Big(q,\varphi(q,1)\Big)$.

If $n^0=1$, we are done. So assume that $n^0>1$. Let $1<n\leq n^0$ and assume that for each $1\leq m<n$, $\varphi(p,m),\varphi(q,m)$ as above have been defined. By the additive representation for $\succsim^p$, we have for each $0\leq m<n$,

$$\begin{aligned}\sum_{i\in\operatorname{supp} p} u\Big(p_i,\varphi(p,m)_i\Big)&=\sum_{i\in\operatorname{supp} p\setminus\{1\}} u\Big(p_i,\varphi(p,m-1)_i\Big)+u(p_1,x_1^1) \\ &=\sum_{i\in\operatorname{supp} p} u\Big(p_i,\varphi(p,m-1)_i\Big)+u(p_1,x_1^1)-u(p_1,x_1^3).\end{aligned} \tag{4}$$

Thus by doing the recursion to the first step,

$$\sum_{i\in\operatorname{supp} p} u\Big(p_i,\varphi(p,n-1)_i\Big)=\sum_{i\in\operatorname{supp} p} u(p_i,x_i^3)+(n-2)\Big(u(p_1,x_1^1)-u(p_1,x_1^3)\Big).$$

By the choice of n^0 and since $n^0>1$,

$$(n^0-1)\Big(u(p_1,x_1^1)-u(p_1,x_1^3)\Big)<\sum_{i\in\operatorname{supp} p\setminus\{1\}} u(p_i,x_i^1)-\sum_{i\in\operatorname{supp} p\setminus\{1\}} u(p_i,x_i^3).$$

Hence,

$$\begin{aligned}\sum_{i\in\operatorname{supp} p} u\Big(p_i,(x_1^1,\varphi(p,n-1)_{-1})_i\Big)&=\sum_{i\in\operatorname{supp} p} u\Big(p_i,\varphi(p,n-1)_i\Big)+u(p_1,x_1^1)-u(p_1,x_1^3) \\ &=\sum_{i\in\operatorname{supp} p} u(p_i,x_i^3)+(n-2+1)\Big(u(p_1,x_1^1)-u(p_1,x_1^3)\Big) \\ &\leq\sum_{i\in\operatorname{supp} p} u(p_i,x_i^3)+(n^0-1)\Big(u(p_1,x_1^1)-u(p_1,x_1^3)\Big) \\ &<\sum_{i\in\operatorname{supp} p} u(p_i,x_i^3)+\sum_{i\in\operatorname{supp} p\setminus\{1\}} u(p_i,x_i^1)-\sum_{i\in\operatorname{supp} p\setminus\{1\}} u(p_i,x_i^3) \\ &=\sum_{i\in\operatorname{supp} p\setminus\{1\}} u(p_i,x_i^1)+u(p_1,x_1^3)=\sum_{i\in\operatorname{supp} p} u\Big(p_i,(x_1^3,x_{-1}^1)_i\Big).\end{aligned} \tag{5}$$

Additionally, since $\varphi(p,n-1)_1 = x^3_1$,

$$\sum_{i\in\operatorname{supp} p} u\Big(p_i,(x^1_1,\varphi(p,n-1)_{-1})_i\Big) > \sum_{i\in\operatorname{supp} p} u\Big(p_i,\varphi(p,n-1)_i\Big).$$

Since X is a connected set as an interval, u is continuous, and $|\operatorname{supp} p| \geq 4$, there exists $\varphi(p,n) \in \mathrm{NDiag}(X^p)$ such that $\varphi(p,n)_1 = x^3_1$, for all $i \in \operatorname{supp} p$, $\varphi(p,n)_i \neq x^1_1$, and

$$\sum_{i\in\operatorname{supp} p} u\Big(p_i,(x^1_1,\varphi(p,n-1)_{-1})_i\Big) = \sum_{i\in\operatorname{supp} p} u\Big(p_i,\varphi(p,n)_i\Big).$$

Thus by the additive representation for $\succsim^p$, $\Big(p,(x^1_1,\varphi(p,n-1)_{-1})\Big) \sim \Big(p,\varphi(p,n)\Big)$.

Second, by the induction assumption and Axiom 7, we have

$$\begin{aligned}&\Big(q,\varphi(q,n{-}1)\Big)\sim\Big(p,\varphi(p,n{-}1)\Big) &(6)\\ &\Rightarrow\Big(q,\Big(x^1_1,\varphi(q,n{-}1)_{-1}\Big)\Big)\sim\Big(p,\Big(x^1_1,\varphi(p,n{-}1)_{-1}\Big)\Big)\overset{(1,5)}{\prec}\Big(p,(x^3_1,\tilde{x}^1_{-1})\Big)\overset{(2)}{\sim}\Big(q,(x^3_1,\tilde{y}^1_{-1})\Big).\end{aligned}$$

By the additive representation and since X is a connected, u is continuous, and $|\operatorname{supp} q| \geq 4$, there exists $\varphi(q,n) \in \mathrm{NDiag}(X^q)$ such that $\varphi(q,n)_1 = x^3_1$, for all $i \in \operatorname{supp} q$, $\varphi(q,n)_i \neq x^1_1$, and

$$\sum_{i\in\operatorname{supp} q} u\Big(q_i,(x^1_1,\varphi(q,n-1)_{-1})_i\Big) = \sum_{i\in\operatorname{supp} q} u\Big(q_i,\varphi(q,n)_i\Big).$$

Thus by the additive representation and (6),

$$\Big(q,\varphi(q,n)\Big) \sim \Big(q,(x^1_1,\varphi(q,n-1)_{-1})\Big) \sim \Big(p,(x^1_1,\varphi(p,n-1)_{-1})\Big) \sim \Big(p,\varphi(p,n)\Big).$$

This completes the induction.

Finally, let $x \in \mathrm{NDiag}(X^p), y \in \mathrm{NDiag}(X^q)$ be such that $(p,x^1) \succsim (p,x),(q,y) \succsim (p,x^3),$. First, we show that

$$(p,x) \succsim (q,y) \iff \sum_{i\in\operatorname{supp} p} u(p_i,x_i) - \sum_{i\in\operatorname{supp} p} u(p_i,x^3_i) \geq \sum_{i\in\operatorname{supp} q} u(q_i,y_i) - \sum_{i\in\operatorname{supp} q} u(q_i,y^3_i).$$

By the additive representations, we have

$$\begin{aligned}&\sum_{i\in\operatorname{supp} p} u(p_i,x^1_i) \geq \sum_{i\in\operatorname{supp} p} u(p_i,x_i) \geq \sum_{i\in\operatorname{supp} p} u(p_i,x^3_i)\\ \text{and}\quad &\sum_{i\in\operatorname{supp} q} u(q_i,y^1_i) \geq \sum_{i\in\operatorname{supp} q} u(q_i,y_i) \geq \sum_{i\in\operatorname{supp} q} u(q_i,y^3_i).\end{aligned}$$

By the definition of n^0, there exist $0 \leq i^x, i^y \leq n^0+1$ such that

$$\sum_{i\in\operatorname{supp} p} u(p_i,x^3_i){+}(i^x{+}1)\Big(u(p_1,x^1_1){-}u(p_1,x^3_1)\Big){>}\sum_{i\in\operatorname{supp} p} u(p_i,x_i){\geq}\sum_{i\in\operatorname{supp} p} u(p_i,x^3_i){+}i^x\Big(u(p_1,x^1_1){-}u(p_1,x^3_1)\Big)$$

and

$$\sum_{i\in\operatorname{supp} q} u(q_i, y_i^3)+(i^y+1)\Big(u(p_1,x_1^1)-u(p_1,x_1^3)\Big)>\sum_{i\in\operatorname{supp} q} u(q_i,y_i)\geq\sum_{i\in\operatorname{supp} q} u(q_i,y_i^3)+i^y\Big(u(p_1,x_1^1)-u(p_1,x_1^3)\Big).$$

For the case $i^x = n^0+1$, define $\varphi(p,n^0+1) = (x_1^1, \varphi(p,n^0+1)_{-1})$, and $\varphi(q,n^0+1) = (x_1^1, \varphi(q,n^0+1)_{-1})$.

First, assume that $i^x \neq i^y$. In this case, we have

$$\sum_{i\in\operatorname{supp} p} u(p_i,x_i) - \sum_{i\in\operatorname{supp} p} u(p_i,x_i^3) \geq \sum_{i\in\operatorname{supp} q} u(q_i,y_i) - \sum_{i\in\operatorname{supp} q} u(q_i,y_i^3) \overset{i^x\neq i^y}{\Longleftrightarrow} i^x > i^y$$

$$\Longleftrightarrow \Big(p,\varphi(p,i^x)\Big) \succ \Big(p,\varphi(p,i^y)\Big) \sim \Big(q,\varphi(q,i^y)\Big) \Longleftrightarrow \Big(p,\varphi(p,i^x)\Big) \succ (q,y) \overset{i^x\neq i^y}{\Longleftrightarrow} (p,x) \succ (q,y),$$

where the first equivalency follows from the definition of i^x, i^y and the assumption that $i^x \neq i^y$, the second equivalency follows from the additive representation for $\succsim^p$, (4), and the definition of φ with an extension to n^0+1, the third equivalency follows from the definition of i^y, and the last one from the assumption that $i^x \neq i^y$ and the definition of i^x.

Next, assume that $i^x = i^y$. If $i^x = n^0+1$, then $(p,x) \sim (p,x^1) \sim (q,y^1) \sim (q,y)$. So assume that $i^x < n^0+1$. By the definitions of i^x, i^y, φ, and continuity of utility, there exist $z_1^x, z_1^y \in [x_1^3, x_1^1]$ such that

$$\sum_{i\in\operatorname{supp} p} u(p_i,x_i) = \sum_{i\in\operatorname{supp} p} u\Big(p_i,(z_1^x,\varphi(p,i^y)_{-1})_i\Big) = \sum_{i\in\operatorname{supp} p} u\Big(p_i,\varphi(p,i^y)_i\Big) + u(p_1,z_1^x) - u(p_1,x_1^3)$$

and since $p_1 = q_1$

$$\sum_{i\in\operatorname{supp} q} u(q_i,y_i) = \sum_{i\in\operatorname{supp} q} u\Big(q_i,(z_1^y,\varphi(q,i^y)_{-1})_i\Big) = \sum_{i\in\operatorname{supp} q} u\Big(q_i,\varphi(q,i^y)_i\Big) + u(p_1,z_1^y) - u(p_1,x_1^3).$$

Thus we have by (4),

$$\sum_{i\in\operatorname{supp} p} u(p_i,x_i) - \sum_{i\in\operatorname{supp} p} u(p_i,x_i^3) \geq \sum_{i\in\operatorname{supp} q} u(q_i,y_i) - \sum_{i\in\operatorname{supp} q} u(q_i,y_i^3)$$

$$\Longleftrightarrow \sum_{i\in\operatorname{supp} p} u(p_i,x_i^3)+(i^x-1)\Big(u(q_1,x_1^1)-u(q_1,x_1^3)\Big)+u(p_1,z_1^x)-u(p_1,x_1^3)-\sum_{i\in\operatorname{supp} p} u(p_i,x_i^3)$$

$$\geq \sum_{i\in\operatorname{supp} q} u(q_i,y_i^3)+(i^y-1)\Big(u(q_1,x_1^1)-u(q_1,x_1^3)\Big)+u(p_1,z_1^y)-u(p_1,x_1^3)-\sum_{i\in\operatorname{supp} q} u(q_i,y_i^3)$$

$$\Longleftrightarrow \quad u(z_1^x) \geq u(z_1^y) \Longleftrightarrow \sum_{i\in\operatorname{supp} p} u\Big(p_i,(z_1^x,\varphi(p,i^y)_{-1})_i\Big) \geq \sum_{i\in\operatorname{supp} p} u\Big(p_i,(z_1^y,\varphi(p,i^y)_{-1})_i\Big)$$

$$\Longleftrightarrow \Big(p,\big(z_1^x,\varphi(p,i^y)_{-1}\big)\Big) \succsim \Big(p,\big(z_1^y,\varphi(p,i^y)_{-1}\big)\Big) \sim \Big(q,\big(z_1^y,\varphi(q,i^y)_{-1}\big)\Big) \Longleftrightarrow (p,x) \succsim (q,y),$$

where the second to last equivalence follows from the additive representation for $\succsim^p$ and the indifference follows from Axiom 7.

Thus we have for all $x \in \mathrm{NDiag}(X^p), y \in \mathrm{NDiag}(X^q)$ such that $(p, x^1) \succsim (p, x), (q, y) \succsim (p, x^3)$,

$$(p, x) \succsim (q, y) \iff \sum_{i \in \operatorname{supp} p} u(p_i, x_i) - \sum_{i \in \operatorname{supp} p} u(p_i, x_i^3) \geq \sum_{i \in \operatorname{supp} q} u(q_i, y_i) - \sum_{i \in \operatorname{supp} q} u(q_i, y_i^3).$$

Since $(p, x^2) \sim (q, y^2)$, by the above and the continuity of utility, we have

$$\sum_{i \in \operatorname{supp} p} u(p_i, x_i^2) - \sum_{i \in \operatorname{supp} p} u(p_i, x_i^3) = \sum_{i \in \operatorname{supp} q} u(q_i, y_i^2) - \sum_{i \in \operatorname{supp} q} u(q_i, y_i^3)$$

and so we can add it to the above equivalencies for the claim. □

The second part shows the local additive representation. Here, we can use continuity to create common probabilities and common prizes for any two induced probability trees to apply the previous lemma. The idea is that if we have two induced probability trees p and q that have a positive probability on the index 1 and zero probability on the index n. Then for a small enough ε_n, we can consider the induced probability trees $p, (p_1 - \varepsilon_n, p_{-1}, \varepsilon_n), (q_1 - \varepsilon_n, q_{-1}, \varepsilon_n)$, and q. Here, p and $(p_1 - \varepsilon_n, p_{-1}, \varepsilon_n)$ share the probabilities p_{-1}, $(p_1 - \varepsilon_n, p_{-1}, \varepsilon_n)$ and $(q_1 - \varepsilon_n, q_{-1}, \varepsilon_n)$ share the probability ε_n, and similarly for $(q_1 - \varepsilon_n, q_{-1}, \varepsilon_n)$ and q. This gives the additive representation locally by applying the previous lemma multiple times.

Lemma 6 Assume that $\succsim$ satisfies Axioms 1-4 and 7 and there exists $u : (0,1) \times X \to \mathbb{R}$ such that for all $p \in \Delta^4(\mathbb{N})$, $(u(p_i, \cdot))_{i \in \operatorname{supp} p}$ is an additive representation for $\succsim^p$.

If $p, q \in \Delta^4(\mathbb{N}), x \in \mathrm{NDiag}(X^p)$, and $y \in \mathrm{NDiag}(X^q)$ are such that $(p, x) \sim (q, y)$, then there exist $P, Q \in \Delta(X)$ such that $P \succ (p, x) \succ Q$ and for all $\tilde{x} \in \mathrm{NDiag}(X^p), \tilde{y} \in \mathrm{NDiag}(X^q)$ such that $P \succsim (p, \tilde{x}), (q, \tilde{y}) \succsim Q$,

$$(p, \tilde{x}) \succsim (q, \tilde{y}) \iff \sum_{i \in \operatorname{supp} p} u(p_i, \tilde{x}_i) - \sum_{i \in \operatorname{supp} p} u(p_i, x_i) \geq \sum_{i \in \operatorname{supp} q} u(q_i, \tilde{y}_i) - \sum_{i \in \operatorname{supp} q} u(q_i, y_i).$$

Proof. Let $c \in X, i^p \in \operatorname{supp} p, i^q \in \operatorname{supp} q$. Since X is open, there exist $x_{i^p}^1, x_{i^p}^2, x_{i^p}^3, x_{i^p}^4 \in X$ such that for all $i \in \operatorname{supp} p, j \in \{1, 2, 3, 4\}$, $x_i \neq x_{i^p}^j$ and $x_{i^p}^1 > x_{i^p}^2 > x_{i^p} > x_{i^p}^3 > x_{i^p}^4$. By Axiom 2,

$$\Big(p, (x_{i^p}^1, x_{-i^p})\Big) \succ \Big(p, (x_{i^p}^2, x_{-i^p})\Big) \succ (p, x) \succ \Big(p, (x_{i^p}^3, x_{-i^p})\Big) \succ \Big(p, (x_{i^p}^4, x_{-i^p})\Big). \tag{7}$$

By Axiom 4, there exists $\alpha^p \in (0,1)$ such that for all $\alpha \in (0, \alpha^p)$,

$$\alpha\Big(p, (c_{i^p}, x_{-i^p})\Big) + (1 - \alpha)\Big(p, (x_{i^p}^1, x_{-i^p})\Big) \succ \Big(p, (x_{i^p}^2, x_{-i^p})\Big) \tag{8}$$

$$\text{and} \qquad \Big(p, (x_{i^p}^3, x_{-i^p})\Big) \succ \alpha\Big(p, (c_{i^p}, x_{-i^p})\Big) + (1 - \alpha)\Big(p, (x_{i^p}^4, x_{-i^p})\Big). \tag{9}$$

Secondly, since X is open and $(p,x)\sim(q,y)$ by (7) and Axiom 3, there exist $y^1_{i^q}, y^2_{i^q}, y^3_{i^q}, y^4_{i^q}, \in X$ such that for all $i \in \operatorname{supp} q$, $j \in \{1,2,3,4\}$ $y^j_{i^q} \neq y_i$, $y^1_{i^q} > y^2_{i^q} > y_{i^q} > y^3_{i^q} > y^4_{i^q}$,

$$\Big(p,(x^2_{i^p},x_{-i^p})\Big) \succ \Big(q,(y^1_{i^q},y_{-i^q})\Big) \succ \Big(q,(y^2_{i^q},y_{-i^q})\Big) \succ (q,y), \tag{10}$$

$$\text{and} \qquad (q,y) \succ \Big(q,(y^3_{i^q},y_{-i^q})\Big) \succ \Big(q,(y^4_{i^q},y_{-i^q})\Big) \succ \Big(p,(x^3_{i^p},x_{-i^p})\Big). \tag{11}$$

By Axiom 4, there exists $\alpha^q \in (0,1)$ such that for all $\alpha \in (0,\alpha^q)$,

$$\Big(q,(y^1_{i^q},y_{-i^q})\Big) \succ \alpha\Big(q,(c_{i^q},y_{-i^q})\Big) + (1-\alpha)\Big(q,(y^2_{i^q},y_{-i^q})\Big) \succ (q,y) \tag{12}$$

$$\text{and} \qquad (q,y) \succ \alpha\Big(q,(c_{i^q},y_{-i^q})\Big) + (1-\alpha)\Big(q,(y^3_{i^q},y_{-i^q})\Big) \succ \Big(q,(y^4_{i^q},y_{-i^q})\Big). \tag{13}$$

Now there exist $\hat\alpha^p \in (0,\alpha^p), \hat\alpha^q \in (0,\alpha^q)$ such that $\hat\alpha^p p_{i^p} = \hat\alpha^q q_{i^q}$.

Let $i^{p^*} \notin \operatorname{supp} p$ and denote $\hat p = (1-\hat\alpha^p)p + \hat\alpha^p(0_{i^p},(p_{i^p})_{i^{p*}},p_{-i^p,i^{p*}})$. Similarly let $i^{q^*} \notin \operatorname{supp} q$ and denote $\hat q = (1-\hat\alpha^q)q + \hat\alpha^q(0_{i^q},(q_{i^q})_{i^{q*}},q_{-i^q,i^{q*}})$. Now using this notation for all $a \in X$, $\tilde x \in \mathrm{NDiag}(X^p)$ such that for all $i \in \operatorname{supp} p$, $\tilde x_i \neq a$, we have $\hat\alpha^p\Big(p,(a_{i^p},\tilde x_{-i^p})\Big) + (1-\hat\alpha^p)(p,\tilde x) = \Big(\hat p,(a_{i^{p*}},\tilde x_{-i^{p*}})\Big)$ and for all $a \in X$, $\tilde y \in \mathrm{NDiag}(X^q)$ such that for all $i \in \operatorname{supp} q$, $\tilde y_i \neq a$, we have $\hat\alpha^q\Big(q,(a_{i^q},\tilde y_{-i^q})\Big) + (1-\hat\alpha^q)(q,\tilde y) = \Big(\hat q,(a_{i^{q*}},\tilde y_{-i^{q*}})\Big)$.

So by (8,10,12),

$$\Big(p,(x^1_{i^p},x_{-i^p})\Big) \succ \Big(\hat p,(c_{i^{p*}},x^1_{i^p},x_{-i^{p*},i^p})\Big) \succ \Big(q,(y^1_{i^q},y_{-i^q})\Big) \succ \Big(\hat q,(c_{i^{q*}},y^2_{i^q},y_{-i^{q*},i^q})\Big) \succ (p,x) \tag{14}$$

and by (9,11,13),

$$(p,x) \succ \Big(\hat q,(c_{i^{q*}},y^3_{i^q},y_{-i^{q*},i^q})\Big) \succ \Big(q,(y^4_{i^q},y_{-i^q})\Big) \succ \Big(\hat p,(c_{i^{p*}},x^4_{i^p},x_{-i^{p*},i^p})\Big) \succ \Big(p,(x^4_{i^p},x_{-i^p})\Big). \tag{15}$$

Let $i^{p,2} \in \operatorname{supp} p, i^p \neq i^{p,2}$ and $i^{q,2} \in \operatorname{supp} q, i^q \neq i^{q,2}$. Since $|\operatorname{supp} p|, |\operatorname{supp} q| \geq 4$ by Axiom 3 and (14,15), there exist $\tilde x^1, \tilde x^2, \tilde y^1, \tilde y^2, \hat x^1, \hat x^2$ such that for $j \in \{1,2\}$

$$(x_{i^{p,2}},\tilde x^j_{-i^{p,2}}) \in \mathrm{NDiag}(X^p), (y_{i^{q,2}},\tilde y^j_{-i^{q,2}}) \in \mathrm{NDiag}(X^q), (c_{i^{p*}},x_{i^{p,2}},\hat x^j_{-i^{p*},i^{p,2}}) \in \mathrm{NDiag}(X^{\hat p}),$$

$$\Big(p,(x_{i^{p,2}},\tilde x^1_{-i^{p,2}})\Big) \sim \Big(q,(y_{i^{q,2}},\tilde y^1_{-i^{q,2}})\Big) \sim \Big(\hat p,(c_{i^{p*}},x_{i^{p,2}},\hat x^1_{-i^{p*},i^{p,2}})\Big) \sim \Big(\hat q,(c_{i^{q*}},y^2_{i^q},y_{-i^{q*},i^q})\Big), \tag{16}$$

and

$$\Big(p,(x_{i^{p,2}},\tilde x^2_{-i^{p,2}})\Big) \sim \Big(q,(y_{i^{q,2}},\tilde y^2_{-i^{q,2}})\Big) \sim \Big(\hat p,(c_{i^{p*}},x_{i^{p,2}},\hat x^2_{-i^{p*},i^{p,2}})\Big) \sim \Big(\hat q,(c_{i^{q*}},y^3_{i^q},y_{-i^{q*},i^q})\Big). \tag{17}$$

Finally, let $z^p \in \mathrm{NDiag}(X^p)$, $z^q \in \mathrm{NDiag}(X^q)$ be such that $\Big(p,(x_{i^{p,2}},\tilde x^1_{-i^{p,2}})\Big) \succsim (p,z^p), (q,z^q) \succsim \Big(p,(x_{i^{p,2}},\tilde x^2_{-i^{p,2}})\Big)$. By (16,17), there exist $z^{\hat p,p} \in \mathrm{NDiag}(X^{\hat p})$ and $z^{\hat q,q} \in \mathrm{NDiag}(X^{\hat q})$ such that $(p,z^p) \sim (\hat p, z^{\hat p,p})$ and $(q,z^q) \sim (\hat q, z^{\hat q,q})$.

Since $\hat{p}_{i^{p,2}} = p_{i^{p,2}}$ and $\hat{q}_{i^{q,2}} = q_{i^{q,2}}$, by Lemma 5, we have

$$\sum_{i \in \operatorname{supp} p} u(p_i, z_i^p) - \sum_{i \in \operatorname{supp} p} u\Big(p_i, (x_{i^{p,2}}, \tilde{x}^1_{-i^{p,2}})_i\Big) = \sum_{i \in \operatorname{supp} \hat{p}} u(\hat{p}_i, z_i^{\hat{p},p}) - \sum_{i \in \operatorname{supp} \hat{p}} u\Big(\hat{p}_i, (c_{i^{p*}}, x_{i^{p,2}}, \hat{x}^1_{-i^{p*}, i^{p,2}})_i\Big)$$

$$\text{and} \sum_{i \in \operatorname{supp} q} u(q_i, z_i^q) - \sum_{i \in \operatorname{supp} q} u\Big(q_i, (y_{i^{q,2}}, \tilde{y}^1_{-i^{q,2}})_i\Big) = \sum_{i \in \operatorname{supp} \hat{q}} u(\hat{q}_i, z_i^{\hat{q},q}) - \sum_{i \in \operatorname{supp} \hat{q}} u\Big(\hat{q}_i, (c_{i^{q*}}, x_{i^{q,2}}, \hat{x}^1_{-i^{q*}, i^{q,2}})_i\Big).$$

Additionally, since $\hat{p}_{i^{p*}} = \hat{\alpha}^p p_{i^p} = \hat{\alpha}^q q_{i^q} = \hat{q}_{i^{q*}}$, by Lemma 5, we have

$$(\hat{p}, z^{\hat{p},p}) \succsim (\hat{q}, z^{\hat{q},q}) \iff \sum_{i \in \operatorname{supp} \hat{p}} u(\hat{p}_i, z_i^{\hat{p},p}) - \sum_{i \in \operatorname{supp} \hat{p}} u\Big(\hat{p}_i, (c_{i^{p*}}, x_{i^{p,2}}, \hat{x}^1_{-(i^{p*}, i^{p,2})})_i\Big)$$
$$\geq \sum_{i \in \operatorname{supp} \hat{q}} u(\hat{q}_i, z_i^{\hat{q},q}) - \sum_{i \in \operatorname{supp} \hat{q}} u\Big(\hat{q}_i, (c_{i^{q*}}, y_{i^{q,2}}, \hat{y}^1_{-(i^{q*}, i^{q,2})})_i\Big).$$

And so by putting these together,

$$(p, z^p) \succsim (q, z^q) \iff$$
$$\sum_{i \in \operatorname{supp} p} u(p_i, z_i^p) - \sum_{i \in \operatorname{supp} p} u\Big(p_i, (x_{i^{p,2}}, \tilde{x}^1_{-i^{p,2}})_i\Big) \geq \sum_{i \in \operatorname{supp} q} u(q_i, z_i^q) - \sum_{i \in \operatorname{supp} q} u\Big(q_i, (y_{i^{q,2}}, \tilde{y}^1_{-i^{q,2}})_i\Big).$$

Since $(p, x) \sim (q, y)$, by the above, we have

$$\sum_{i \in \operatorname{supp} p} u(p_i, x_i) - \sum_{i \in \operatorname{supp} p} u\Big(p_i, (x_{i^{p,2}}, \tilde{x}^1_{-i^{p,2}})_i\Big) = \sum_{i \in \operatorname{supp} q} u(q_i, y_i) - \sum_{i \in \operatorname{supp} q} u\Big(q_i, (y_{i^{q,2}}, \tilde{y}^1_{-i^{q,2}})_i\Big)$$

so we can subtract it from the both sides of the above equivalencies for the final claim. □

A.4 Extending the Local Representation to a Global Representation

First, we define an information function on lotteries that only depends on the induced probability tree.

Definition A function $c : \Delta(X) \to \mathbb{R}$ is *an information function* if for all $P, Q \in \Delta(X)$ with the same induced probability tree, $c(P) = c(Q)$.

By Mononen (2021), we can extend the previous local representation to a global representation by adding induced probability tree-dependent constant.

Lemma 7 Assume that $\succsim$ satisfies Axioms 1-4 and 7 and there exists $u : (0,1) \times X \to \mathbb{R}$ such that for all $p \in \Delta^4(\mathbb{N})$, $(u(p_i, \cdot))_{i \in \operatorname{supp} p}$ is an additive representation for $\succsim^p$. Then there exists an information function $c : \Delta(X) \to \mathbb{R}$ such that for all simple lotteries P, Q with $|\operatorname{supp} P|, |\operatorname{supp} Q| \geq 4$,

$$P \succsim Q \iff \sum_{x \in \operatorname{supp} P} u(P(x), x) + c(P) \geq \sum_{x \in \operatorname{supp} Q} u(Q(x), x) + c(Q).$$

A.5 Characterizing Prospect Theory

In this section, we replace Axiom 7 with Axioms 5 and 6 and show that under these stronger axioms the representation from Lemma 7 simplifies to prospect theory.

First, we show that under Axiom 5, we can strengthen Proposition 4 for prospect theory within induced probability tree.

Proposition 8 Assume that $\succsim$ satisfies Axioms 1-5. Then for all $a \in (0,1)$ there exist $w : [0,1] \to \mathbb{R}$ and a continuous and strictly increasing $u(\cdot)$ such that $u(0) = 0$ and for all $x, y \in \text{NDiag}(X^p)$,

$$(p, x) \succsim (p, y) \iff \sum_{i \in \operatorname{supp} p} w(p_i)u(x_i) \geq \sum_{i \in \operatorname{supp} p} w(p_i)u(y_i).$$

Proof. By Proposition 4, there exists $u : (0,1) \times X \to \mathbb{R}$ such that for all $p \in \Delta^4(\mathbb{N})$, $(u(p_i, \cdot))_{i \in \operatorname{supp} p}$ is an additive representation for $\succsim^p$ with for all $a \in (0,1)$, $u(a, 0) = 0$ by the normalization. Let $p \in \Delta^4(\mathbb{N})$. We show that there exists $w : [0,1] \to \mathbb{R}$, $\theta : (0,1) \to \mathbb{R}$, and a continuous and strictly increasing u such that for each p_i and x_i, $u(p_i, x_i) = w(p_i)u(x_i)$. Directly from Axiom 5, if

$$u(p_i, x) - u(p_i, y) \geq u(p_i, z) - u(p_i, m), \text{ then } u(p_j, x) - u(p_j, y) \geq u(p_j, z) - u(p_i, m).$$

Thus the result follows symmetrically to Wakker (1984). □

Next, we show that Axiom 6 makes the information function in Lemma 7 a constant.

Lemma 9 If $\succsim$ satisfies Axioms 1-6, then there exist $w : [0,1] \to \mathbb{R}$ and a continuous, strictly increasing, $u : X \to \mathbb{R}$ such that for all simple lotteries P, Q with $|\operatorname{supp} P|, |\operatorname{supp} Q| \geq 4$,

$$P \succsim Q \iff \sum_{x \in \operatorname{supp} P} w(P(x))u(x) \geq \sum_{x \in \operatorname{supp} Q} w(Q(x))u(x).$$

Proof. By Lemma 7 and Proposition 8, there exist $w : [0,1] \to \mathbb{R}$, a continuous, strictly increasing, $u : X \to \mathbb{R}$ and an information function $c : \Delta(X) \to \mathbb{R}$ such that for all simple lotteries P, Q with $|\operatorname{supp} P|, |\operatorname{supp} Q| \geq 4$,

$$P \succsim Q \iff \sum_{x \in \operatorname{supp} P} w(P(x))u(x) + c(P) \geq \sum_{x \in \operatorname{supp} Q} w(Q(x))u(x) + c(Q).$$

We show that for all $P, Q \in \Delta(X)$ with $|\operatorname{supp} P| \geq 4$, $c(P) = c(Q)$. Assume, per contra, that there exist $p, q \in \Delta^4(\mathbb{N})$ with $c(p) > c(q)$. Since $u(0) = 0$ by the normalization, by the continuity of u, there exist $x \in \mathrm{NDiag}(X^p)$ and $y \in \mathrm{NDiag}(X^q)$ such that for all $i \in \operatorname{supp} p, j \in \operatorname{supp} q$, $y_j > 0 > x_i$, and

$$-\sum_{i \in \operatorname{supp} p} w(p_i)u(x_i), \sum_{j \in \operatorname{supp} q} w(q_j)u(y_j) < \frac{1}{2}(c(p) - c(q)).$$

Thus,

$$\sum_{j \in \operatorname{supp} q} w(q_j)u(y_j) + c(q) < \frac{1}{2}(c(p) + c(q)) < \sum_{i \in \operatorname{supp} i} w(p_i)u(x_j) + c(p).$$

However, by Axiom 6, $(p, x) \succsim \delta_0 \succsim (q, y)$ that is a contradiction with the representation. □

Finally, we show that the probability weighting function w is continuous and we can extend the representation for all the lotteries. This shows sufficiency for Theorem 1.

Proof of Theorem 1. Necessity of the axioms is standard and omitted. We show the sufficiency of the axioms.

By Lemma 9, there exist $w : [0,1] \to \mathbb{R}$ and a continuous, strictly increasing, $u : X \to \mathbb{R}$ such that for all simple lotteries P, Q with $|\operatorname{supp} P|, |\operatorname{supp} Q| \geq 4$,

$$P \succsim Q \iff \sum_{x \in \operatorname{supp} P} w(P(x))u(x) \geq \sum_{x \in \operatorname{supp} Q} w(Q(x))u(x).$$

We show first that $\lim_{p \to 0} w(p) = 0$. Let $\varepsilon > 0$ and let $x^1, x^2, x^3 \in X$ with $x^1 > x^2 > x^3 > 0$. Let $P \in \Delta(X)$ with $|\operatorname{supp} P| \geq 4$ and $0 < \sum_{x \in \operatorname{supp} P} w(P(x))u(x) < 3u(x^3)\varepsilon$. By the continuity u, there exists $Q \in \Delta(X)$ with $|\operatorname{supp} Q| \geq 4$ such that for all $x \in \operatorname{supp} Q$, $x > 0$ and $\sum_{x \in \operatorname{supp} P} w(P(x))u(x) > \sum_{x \in \operatorname{supp} Q} w(Q(x))u(x)$. Thus $P \succ \delta_0$. By Axiom 4, there exists $\alpha^* \in (0,1)$ such that for all $\alpha \in (0, \alpha^*)$,

$$P \succsim \alpha\Big(\frac{1}{3}\delta_{x^1} + \frac{1}{3}\delta_{x^2} + \frac{1}{3}\delta_{x^3}\Big) + (1-\alpha)\delta_0.$$

Thus by the representation, for all $\alpha \in (0, \alpha^*)$

$$3u(x^3)\varepsilon > \sum_{x \in \operatorname{supp} P} w(P(x))u(x) \geq w(\frac{\alpha}{3})u(x^1) + w(\frac{\alpha}{3})u(x^2) + w(\frac{\alpha}{3})u(x^3) > 3w(\frac{\alpha}{3})u(x^3)$$

and so for all $\alpha \in (0, \frac{\alpha^*}{3})$, $\varepsilon > w(\alpha)$. Since $\varepsilon > 0$ was arbitrary, this shows that $\lim_{p \to 0} w(p) = 0$.

Next, we show that there exists $c \in \mathbb{R}$ such that $\lim_{p \to 1} w(p) = c$. Let $(r_n)_{n \in \mathbb{N}} \subset (0,1)$ be such that $r_n \to 1$ as $n \to \infty$ and $\lim_{n \to \infty} w(r_n)$ exists and $(w(r_n))_{n \in \mathbb{N}}$ is a monotone sequence. Assume w.l.o.g. that $(w(r_n))_{n \in \mathbb{N}}$ is an increasing sequence. By possibly taking a subsequence, assume w.l.o.g. that $(w(\frac{1-r_n}{3}))_{n \in \mathbb{N}}$ is a monotone sequence. By above, $(w(\frac{1-r_n}{3}))_{n \in \mathbb{N}}$ is a weakly decreasing sequence.

Denote $\lim_{n\to\infty} w(r_n) = c$. Let $\tilde{\varepsilon} > 0$ and let $x^1 \in X$ with $x^1 > 0$. Let $\varepsilon < \tilde{\varepsilon} u(x^1)$ be such that $[u(x^1), (u(x^1)+\varepsilon)(1+\varepsilon)] \subseteq u(X)$. There exists $n^0 \in \mathbb{N}$ such that $w(r_{n^0}) > \frac{c}{1+\varepsilon}$ and $\frac{\varepsilon}{4} > w(\frac{1-r_{n^0}}{3})$. Now, by the continuity of u and the choice of ε, there exists $x^2 > x^1$ and $x^3, x^4 > 0$ such that $1 > u(x^3), u(x^4)$ and

$$cu(x^1) + \varepsilon > w(r_{n^0})u(x^2) + w\Big(1 - \frac{r_{n^0}}{3}\Big)u(x^3) + w\Big(1 - \frac{r_{n^0}}{3}\Big)u(x^4) > cu(x^1) + \frac{\varepsilon}{2}.$$

For all $n > n^0$,

$$\begin{aligned} &w(r_n)u(x^1) + w\Big(1 - \frac{r_n}{3}\Big)u(x^3) + w\Big(1 - \frac{r_n}{3}\Big)u(x^4) \\ &< cu(x^1) + w\Big(1 - \frac{r_n}{3}\Big)u(x^3) + w\Big(1 - \frac{r_n}{3}\Big)u(x^4) < cu(x^1) + \frac{\varepsilon}{2}. \end{aligned}$$

Thus, by the representation, there exists $P \in \Delta(X)$ such that for all $n \in \mathbb{N}$

$$r_{n^0}\delta_{x^2} + \frac{1-r_{n^0}}{3}\delta_{x^3} + \frac{1-r_{n^0}}{3}\delta_{x^4} + \frac{1-r_{n^0}}{3}\delta_0 \succ P \succ r_n\delta_{x^1} + \frac{1-r_n}{3}\delta_{x^3} + \frac{1-r_n}{3}\delta_{x^4} + \frac{1-r_n}{3}\delta_0.$$

By Axiom 4,

$$r_{n^0}\delta_{x^2} + \frac{1-r_{n^0}}{3}\delta_{x^3} + \frac{1-r_{n^0}}{3}\delta_{x^4} + \frac{1-r_{n^0}}{3}\delta_0 \succ \delta_{x^1}.$$

By Axiom 4, there exists $\alpha^+ \in (0,1)$ such that for all $\alpha \in (\alpha^+, 1)$

$$r_{n^0}\delta_{x^2} + \frac{1-r_{n^0}}{3}\delta_{x^3} + \frac{1-r_{n^0}}{3}\delta_{x^4} + \frac{1-r_{n^0}}{3}\delta_0 \succ \alpha\delta_{x^1} + \frac{1-\alpha}{3}\delta_{x^3} + \frac{1-\alpha}{3}\delta_{x^4} + \frac{1-\alpha}{3}\delta_0.$$

By the representation, for all $\alpha \in (\alpha^+, 1)$,

$$\begin{aligned} &cu(x^1) + \varepsilon > w(r_{n^0})u(x^2) + w\Big(1 - \frac{r_{n^0}}{3}\Big)u(x^3) + w\Big(1 - \frac{r_{n^0}}{3}\Big)u(x^4) \\ &> w(\alpha)u(x^1) + w\Big(1 - \frac{\alpha}{3}\Big)u(x^3) + w\Big(1 - \frac{\alpha}{3}\Big)u(x^4) > w(\alpha)u(x^1) \end{aligned}$$

and so for all $\alpha \in (\alpha^+, 1)$, $\tilde{\varepsilon} > \frac{\varepsilon}{u(x^1)} > w(\alpha) - c$.

Symmetrically, by repeating the argument for $0 > x^5, x^6, x^7$ and $x^6 > x^7$ such that $cu(x^1) - \varepsilon < w(r_{n^0})u(x^1) + w\Big(\frac{r_{n^0}}{3}\Big)u(x^5) + w\Big(\frac{r_{n^0}}{3}\Big)u(x^7)$, we get that there exists $\alpha^- < 1$ such that for all $\alpha \in (\alpha^-, 1)$, $\tilde{\varepsilon} > \frac{\varepsilon}{u(x^1)} > c - w(\alpha)$.

Since $\tilde{\varepsilon} > 0$ was arbitrary, this shows that $\lim_{p\to 1} w(p) = c$. Thus by scaling the representation, we can assume w.l.o.g. that $\lim_{p\to 1} w(p) = 1$.

Next, we show that w is continuous in $(0,1)$. Let $x, y, z \in X$ with $x > y > z > 0$. Let $p \in (0,1)$ and let $\varepsilon > 0$. By the representation and the continuity of u, there exist P and Q

with $|\operatorname{supp} P|, |\operatorname{supp} Q| \geq 4$, and

$$w(p)u(x) + w\Big(\frac{1-p}{3}\Big)u(y) + w\Big(\frac{1-p}{3}\Big)u(z) + \frac{\varepsilon}{u(x)} > \sum_{x \in \operatorname{supp} P} w(P(x))u(x)$$
$$> w(p)u(x) + w\Big(\frac{1-p}{3}\Big)u(y) + w\Big(\frac{1-p}{3}\Big)u(z)$$
$$> \sum_{x \in \operatorname{supp} Q} w(Q(x))u(x) > w(p)u(x) + w\Big(\frac{1-p}{3}\Big)u(y) + w\Big(\frac{1-p}{3}\Big)u(z) - \frac{\varepsilon}{u(x)}.$$

By the representation, $P \succ p\delta_x + \frac{1-p}{3}\delta_y + \frac{1-p}{3}\delta_z + \frac{1-p}{3}\delta_0 \succ Q$. By applying Axiom 4 twice, there exists $0 < \beta < p, 1-p$ such that for all $\alpha \in (p-\beta, p+\beta)$,

$$P \succ \alpha\delta_x + \frac{1-p}{3}\delta_y + \frac{1-p}{3}\delta_z + (1 - \alpha - 2\frac{1-p}{3})\delta_0 \succ Q$$

By the representation for all $\alpha \in (p-\beta, p+\beta)$,

$$w(p)u(x) + w\Big(\frac{1-p}{3}\Big)u(y) + w\Big(\frac{1-p}{3}\Big)u(z) + \frac{\varepsilon}{u(x)} > w(\alpha)u(x) + w\Big(\frac{1-p}{3}\Big)u(y) + w\Big(\frac{1-p}{3}\Big)u(z)$$
$$> w(p)u(x) + w\Big(\frac{1-p}{3}\Big)u(y) + w\Big(\frac{1-p}{3}\Big)u(z) - \frac{\varepsilon}{u(x)}$$

and so for all $\alpha \in (p-\beta, p+\beta)$, $\varepsilon > w(\alpha) - w(p) > -\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows the continuity of w.

Next, we show that for all lotteries (p, x) and (q, y)

$$(p, x) \succ (q, y) \iff \sum_{i \in \operatorname{supp} p} w(p_i)u(x_i) > \sum_{i \in \operatorname{supp} q} w(q_i)u(y_i).$$

First, assume that $(p, x) \succ (q, y)$. By Axiom 3 and since X is open, there exist $\tilde{x} \in \operatorname{NDiag}(X^p), \tilde{y} \in \operatorname{NDiag}(X^p)$ such that for each $i \in \operatorname{supp} p$, $x_i > \tilde{x}_i$ and $j \in \operatorname{supp} q$, $\tilde{y}_j > y_j$ with $(p, \tilde{x}) \succ (q, \tilde{y})$. Let $r \in \Delta(\mathbb{N})$ and $z \in \operatorname{NDiag}(X^r)$ be such that for all $i \in \operatorname{supp} r$, $j \in \operatorname{supp} p$, $k \in \operatorname{supp} q$, $x_j \neq z_i \neq y_k$. By the continuity of w, monotonicity of u, and Axiom 4, there exists $\alpha \in (0, 1)$ such that

$$\sum_{i \in \operatorname{supp} p} w(p_i)u(x_i) > \sum_{i \in \operatorname{supp} p} w(\alpha p_i)u(\tilde{x}_i) + \sum_{j \in \operatorname{supp} r} w\Big((1-\alpha)r_i\Big)u(z_i),$$
$$\sum_{i \in \operatorname{supp} p} w(\alpha p_i)u(\tilde{y}_i) + \sum_{j \in \operatorname{supp} r} w\Big((1-\alpha)r_i\Big)u(z_i) > \sum_{i \in \operatorname{supp} q} w(p_i)u(y_i),$$

and $\alpha(p, \tilde{x}) + (1-\alpha)(r, z) \succ \alpha(q, \tilde{y}) + (1-\alpha)(r, z)$. Thus by the representation

$$\sum_{i \in \operatorname{supp} p} w(\alpha p_i)u(\tilde{x}_i) + \sum_{j \in \operatorname{supp} r} w\Big((1-\alpha)r_i\Big)u(z_i) > \sum_{i \in \operatorname{supp} p} w(\alpha p_i)u(\tilde{y}_i) + \sum_{j \in \operatorname{supp} r} w\Big((1-\alpha)r_i\Big)u(z_i)$$

and so by the strict monotonicity of u and continuity of w, $\sum_{i \in \operatorname{supp} p} w(p_i)u(x_i) > \sum_{i \in \operatorname{supp} q} w(p_i)u(y_i)$.

Second, assume that $\sum_{i \in \operatorname{supp} p} w(p_i)u(x_i) > \sum_{i \in \operatorname{supp} q} w(q_i)u(y_i)$. Since u is continuous and

X is open, there exist $\tilde{x} \in \mathrm{NDiag}(X^p), \tilde{y} \in \mathrm{NDiag}(X^q)$ such that for each $i \in \operatorname{supp} p$, $x_i > \tilde{x}_i$ and $j \in \operatorname{supp} q$, $\tilde{y}_j > y_j$ with $\sum_{i \in \operatorname{supp} p} w(p_i)u(\tilde{x}_i) > \sum_{i \in \operatorname{supp} q} w(p_i)u(\tilde{y}_i)$.

Let $r \in \Delta(\mathbb{N})$ and $z \in \mathrm{NDiag}(X^r)$ with for all $i \in \operatorname{supp} r$, $j \in \operatorname{supp} p$, $k \in \operatorname{supp} q$, $x_j \neq z_i \neq y_k$. By the continuity of w, there exists $\alpha^* \in (0,1)$ such that for all $\alpha \in (\alpha^*, 1)$

$$\sum_{i \in \operatorname{supp} p} w(\alpha p_i)u(\tilde{x}_i) + \sum_{j \in \operatorname{supp} r} w\Big((1-\alpha)r_i\Big)u(z_i) > \sum_{i \in \operatorname{supp} q} w(\alpha q_i)u(\tilde{y}_i) + \sum_{j \in \operatorname{supp} r} w\Big((1-\alpha)r_i\Big)u(z_i).$$

By the representation for all $\alpha \in (\alpha^*, 1)$, $\alpha(p, \tilde{x}) + (1-\alpha)(r, z) \succsim \alpha(q, \tilde{y}) + (1-\alpha)(r, z)$. By Axioms 2 and 4, $(p, x) \succ (p, \tilde{x}) \succsim (q, \tilde{y}) \succ (q, y)$. $\square$

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