Infinite-dimensional nonlinear stationary Fokker–Planck–Kolmogorov equations

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Dedicated to Leonard Gross on the occasion of his 95th birthday

Abstract. We prove existence of a probability solution to the nonlinear stationary Fokker–Planck–Kolmogorov equation on an infinite dimensional space with a centered Gaussian measure γ with a unit diffusion operator and a drift of the form $-x+v(p,x)$, where v is a bounded mapping with values in the Cameron–Martin space H of γ and v is defined on the space $E\times X$, where is E is the subset of $L^2(\gamma)$ consisting of probability densities. The equation has the form $L^*_{b(p,\bullet)}(p\cdot\gamma)=0$ with $L_{b(p,\bullet)}\varphi=\Delta_H\varphi+(b(p,\bullet),D_H\varphi)_H$, so that the drift coefficient depends on the unknown solution, which makes the equation nonlinear. This dependence is assumed to satisfy a suitable continuity condition. This result is applied to drifts of Vlasov type defined by means of the convolution of a vector field with the solution. In addition, we consider a more general situation where only the components of v are uniformly bounded and prove the existence of a probability solution under some stronger continuity condition on the drift.

Keywords: stationary Fokker–Planck–Kolmogorov equation, nonlinear Fokker–Planck–Kolmogorov equation, Gaussian measure

1. Introduction

Leonard Gross, one of the pioneers of infinite-dimensional analysis, initiated his program of the study of infinite-dimensional harmonic analysis in his seminal papers [**16**], [**17**], and [**18**]. In the subsequent years, this direction was developed by Gross himself, his students, including Piech, Kuo, and Gordina (see, e.g., [**15**], [**22**], [**23**], [**24**], and [**25**]), and many followers all over the world. Part of Gross's program was the investigation of elliptic equations on infinite-dimensional spaces, see, e.g., [**11**], [**12**], [**13**], and [**14**] (of course, there is also a parabolic counterpart which is not discussed here). As in the finite-dimensional case, several different types of elliptic equations arise: direct equations such as $Lf=\Delta f+\langle b,\nabla f\rangle=0$, divergence type equations, and double-divergence type equations or adjoint equations, which in case of a constant second order coefficients look like $\Delta f-\operatorname{div}(fb)=0$. The latter equation is naturally defined for measures as

$$L^*\mu=0,$$

which in the finite-dimensional case is understood as the identity

$$\int_{\mathbb{R}^n} L\varphi\,d\mu=0,\quad \varphi\in C_0^\infty(\mathbb{R}^n),$$

and in such a form is called the stationary Fokker–Planck–Kolmogorov equation. This equation is meaningful if b is locally integrable with respect to μ, in particular, if b is locally bounded.

A nonlinear equation arises when the drift b depends on the unknown solution μ. For example, a Vlasov-type equation corresponds to the drift

$$b(\mu, x) = \int_{\mathbb{R}^n} b_0(x-y)\,\mu(dy).$$

For a recent survey of nonlinear Fokker–Planck–Kolmogorov equations see [**10**]. In this paper, we study nonlinear stationary Fokker–Planck–Kolmogorov equations on infinite-dimensional spaces. Such equations are defined similarly, which is discussed in Section 2.

Our main result (stated and proved in Section 3) gives the existence of a probability to the infinite-dimensional nonlinear stationary equation with the drift

$$b(\mu, x) = -x + v(\mu, x),$$

where v takes values in the Cameron–Martin space H of a centered Radon Gaussian measure γ on a locally convex space X and is uniformly bounded with respect to the Cameron–Martin norm on H and is continuous with respect to μ is a suitable sense. A solution is constructed in the set of probability measures with densities from $L^2(\gamma)$. It is also shown that if the field v_u depends Borel measurably on a parameter u from a complete separable metric space U, then one can select a solution p_u that is Borel measurable in u. Finally, we consider a more general case where only the components of v are uniformly bounded. However, in order to ensure the existence of a probability solution in this case we impose some stronger continuity condition on v.

2. Notation, terminology and auxiliary results

Let γ be a centered Radon Gaussian measure on a locally convex space X (see, e.g., [**2**]). The space of continuous linear functionals on X is denoted by X^*. Without loss of generality one can assume that $X = \mathbb{R}^\infty$ is the countable power of the real line and γ is the countable power of the standard Gaussian measure on the real line. In that case the dual space X^* can be identified with the space $\mathbb{R}^\infty_0$ of finite sequences. This measure γ is called the standard Gaussian measure on $\mathbb{R}^\infty$.

Let X^*_γ be the closure of X^* in $L^2(\gamma)$. For the standard Gaussian measure γ on $\mathbb{R}^\infty$ the space X^*_γ can be identified with the standard Hilbert space l^2: each element (h_n) of l^2 generates the element $\widehat{h}(x) = \sum_{n=1}^\infty h_n x_n$ of X^*_γ, and conversely every element of X^*_γ admits such representation.

The Cameron–Martin space $H = H(\gamma)$ of γ is the space of all vectors h with finite norm

$$|h|_H = \sup\Bigl\{ l(h)\colon l \in X^*, \int_X l^2\, d\gamma \le 1 \Bigr\}.$$

For every element $h \in H(\gamma)$ there is an element $\widehat{h} \in X^*_\gamma$ such that

$$l(h) = \int_X \widehat{h} l\, d\gamma \quad \forall\, l \in X^*.$$

The norm on $H = H(\gamma)$ indicated above is generated by the inner product

$$(u, v)_H = \int_X \widehat{u}\widehat{v}\, d\gamma.$$

The mapping $h \mapsto \widehat{h}$ is a linear isometry of $H(\gamma)$ and X^*_γ and the inverse isometry is denoted by j_H. Thus, $j_H(\widehat{h}) = h$ and

$$(j_H(f), j_H(g))_H = (f, g)_{L^2(\gamma)} \quad \forall\, f, g \in X^*_\gamma.$$

For the standard Gaussian measure γ on $\mathbb{R}^\infty$ the Cameron–Martin space coincides with l^2 and $h \mapsto \widehat{h}$ is the identity mapping when X^*_γ is identified with l^2 as explained above.

It is known that the Cameron–Martin space is a separable Hilbert space. In addition, the measure γ is concentrated on a countable union of metrizable compact sets, so one can assume below that X is such a space. Moreover, if H is infinite-dimensional, then there is a Borel linear isomorphism between γ and the standard Gaussian measure γ_∞ in the following sense: there are Borel linear subspaces $X_1 \subset X$ and $X_2 \subset \mathbb{R}^\infty$ with $\gamma(X_1) = \gamma_\infty(X_2) = 1$ and one-to-one Borel linear operator $J\colon X_1 \to X_2$ with Borel J^{-1} such that J takes γ to γ_∞ and $J\colon H \to l^2$ is an isometry. It follows that the problem we discuss below reduces to the case of $\mathbb{R}^\infty$ with the standard Gaussian measure. The readers who prefer to deal with coordinate representations can assume that we consider this case.

It is always possible to find an orthonormal basis $\{e_n\}$ in $H(\gamma)$ such that $\widehat{e}_n \in X^*$. For all $h \in H$ and $l \in X^*$ we have

$$l(h) = (h, j_H(l))_H = \sum_{n=1}^\infty (h, e_n)_H (j_H(l), e_n)_H = \sum_{n=1}^\infty \widehat{e}_n(h) l(e_n). \tag{2.1}$$

According to Tsirelson's theorem (see [**2**]), for any orthonormal basis $\{e_n\}$ in $H(\gamma)$, we have

$$x = \sum_{n=1}^\infty \widehat{e}_n(x) e_n \quad \gamma\text{-a.e.},$$

where the series converges in X. In particular, for every $l \in X^*$ we have

$$l(x) = \sum_{n=1}^\infty \widehat{e}_n(x) l(e_n) \quad \gamma\text{-a.e.} \tag{2.2}$$

Let $\mathcal{FC}$ be the space of functions on X of the form

$$\varphi(x) = \psi(l_1(x), \dots, l_n(x)), \quad \psi \in C_b^\infty(\mathbb{R}^n),\ l_i \in X^*.$$

Such functions will be called smooth cylindrical. Any function $\varphi \in \mathcal{FC}$ of this form has bounded partial derivatives $\partial_h \varphi$ for all $h \in X$ and

$$\partial_h \varphi(x) = \lim_{t \to 0} \frac{\varphi(x + th) - \varphi(x)}{t} = \sum_{i=1}^n \partial_{x_j} \psi(l_1(x), \dots, l_n(x)) l_j(h).$$

Consequently,

$$\partial_h^2 \varphi = \sum_{i,j=1}^n \partial_{x_j} \partial_{x_i} \psi(l_1, \dots, l_n) l_i(h) l_j(h).$$

Suppose that $v\colon X \to H$ is a Borel mapping and

$$b(x) = -x + v(x).$$

Let us fix an orthonormal basis $\{e_n\}$ in $H(\gamma)$ such that $\widehat{e}_n \in X^*$. Then we can define the operator

$$L_b\varphi := \sum_{i=1}^{\infty} [\partial_{e_i}^2\varphi + \widehat{e}_i(b(x))\partial_{e_i}\varphi(x)]$$

defined on $\mathcal{FC}$ and taking values in $L^2(\gamma)$. Indeed, writing φ as above, we have

$$\begin{aligned}\partial_{e_i}^2\varphi &+ \widehat{e}_i(b(x))\partial_{e_i}\varphi(x)\\ &= \sum_{j,k\le n} \partial_{x_k}\partial_{x_j}\psi(l_1,\dots,l_n)l_k(e_i)l_j(e_i) + \sum_{j\le n} \widehat{e}_i(b(x))\partial_{x_j}\psi(l_1,\dots,l_n)l_j(e_i),\end{aligned}$$

which gives

$$\begin{aligned}L_b\varphi = \sum_{j,k\le n} \partial_{x_k}\partial_{x_j}\psi(l_1,\dots,l_n) \sum_{i=1}^{\infty} l_k(e_i)l_j(e_i)\\ + \sum_{j\le n} \partial_{x_j}\psi(l_1,\dots,l_n) \sum_{i=1}^{\infty} \widehat{e}_i(b(x))l_j(e_i),\end{aligned}$$

where

$$\sum_{i=1}^{\infty} l_k(e_i)l_j(e_i) = (j_H(l_k), j_H(l_j))_H,$$

$$\sum_{i=1}^{\infty} \widehat{e}_i(b(x))l_j(e_i) = -\sum_{i=1}^{\infty} \widehat{e}_i(x)l_j(e_i) + \sum_{i=1}^{\infty} \widehat{e}_i(v(x))l_j(e_i).$$

According to (2.1) and (2.2) the latter equals γ-a.e.

$$-l_j(x) + l_j(v(x)) = l_j(b(x)).$$

Therefore,

$$L_b\varphi = \sum_{j,k\le n} \partial_{x_k}\partial_{x_j}\psi(l_1,\dots,l_n)(j_H(l_k), j_H(l_j))_H + \sum_{j\le n} l_j(b(x)).$$

Thus, $L_b\varphi \in L^2(\gamma)$ and L_b does not depend on our choice of $\{e_n\}$ with the indicated properties.

Recall that a Borel probability measure is called Radon if for every Borel set B we have $\mu(B) = \sup \mu(K)$, where sup is taken over compact subsets of B.

A Radon probability measure μ absolutely continuous with respect to γ satisfies the stationary Fokker–Planck–Kolmogorov equation

$$L_b^*\mu = 0 \tag{2.3}$$

if $l(b) \in L^1(\mu)$ for all $l \in X^*$ and

$$\int_X L_b\varphi \, d\mu = 0 \quad \forall \varphi \in \mathcal{FC}. \tag{2.4}$$

It is worth noting that (2.3) can be interpreted in a weaker sense: we fix an orthonormal basis $\{e_n\}$ in $H(\gamma)$ such that $\widehat{e}_n \in X^*$, consider the class $\mathcal{FC}_{\{e_n\}}$ of smooth cylindrical functions of the form indicated above defined by means of the sequence of functionals $l_n = \widehat{e}_n$ and define (2.3) by means of the identity (2.4) on $\mathcal{FC}_{\{e_n\}}$. However, it is known (see [**7**] or [**2**, Theorem 7.5.6]) that if v is bounded, then any Borel probability measure μ with $X^* \subset L^2(\mu)$ satisfying the equation $L_b^*\mu = 0$ in this weaker sense is absolutely continuous with respect to γ, moreover, its Radon–Nikodym density belongs to $L^2(\gamma)$ (see [**4**]). Note also that in this case $\varrho = d\mu/d\gamma$ belongs to the Gaussian Sobolev class $W^{1,1}(\gamma)$ and

$$\int_X \frac{|D_H\varrho|_H^2}{\varrho}\,d\gamma \le \int_X |b|_H^2\,d\gamma.$$

A more general situation is studied in [**9**], where the following theorem is proved. Let us consider the class $\mathcal{FC}_{0,\{e_n\}}$ of smooth cylindrical functions of the form

$$\varphi(x) = \psi(l_1(x), \ldots, l_n(x))$$

with $l_n = \widehat{e}_n$ and functions $\psi \in C_0^\infty(\mathbb{R}^n)$; unlike $\mathcal{FC}$, this class is not a linear space. If $X = \mathbb{R}^\infty$ and $H = l^2$ with its natural basis $\{e_n\}$, then $l_n = \widehat{e}_n$ is the n-coordinate function and $\mathcal{FC}_{0,\{e_n\}}$ is just the union of all $C_0^\infty(\mathbb{R}^n)$.

A Borel measure μ is called a solution to the equation $L_b^*\mu = 0$ with respect to $\mathcal{FC}_{0,\{e_n\}}$ if $l_n(v) \in L^1(\mu)$ for all n and (2.4) holds for all $\varphi \in \mathcal{FC}_{0,\{e_n\}}$. Although $\mathcal{FC}_{0,\{e_n\}}$ is not a linear space, an advantage of using this class is that for any function $\varphi \in \mathcal{FC}_{0,\{e_n\}}$ the functions $l_j\partial_{e_j}\varphi$ are bounded, so in the case of bounded $l_j(v)$ the functions $L_b\varphi$ are also bounded and belong to $L^1(\mu)$.

THEOREM 2.1. *Let μ be a Borel probability measure on X such that $|v|_H \in L^1(\mu)$ and $L_b^*\mu = 0$ with respect to $\mathcal{FC}_{0,\{e_n\}}$, where $b(x) = -x + v(x)$. Then μ is absolutely continuous with respect to γ and for its Radon–Nikodym $f := d\mu/d\gamma$ we have*

$$\int_X f\bigl(\log(f+1)\bigr)^\alpha\,d\gamma \le C(\alpha)\Bigl[1 + \bigl\||v|_H\bigr\|_{L^1(\mu)}\Bigl(\log\bigl(1+\bigl\||v|_H\bigr\|_{L^1(\mu)}\bigr)\Bigr)^\alpha\Bigr] \tag{2.5}$$

for every $\alpha < 1/4$, where $C(\alpha)$ is a number depending only on α.

It follows from this result that the solution with respect to $\mathcal{FC}_{0,\{e_n\}}$ will be also a solution with respect to $\mathcal{FC}$, because any function $f \in \mathcal{FC}$ can be approximated in any $L^p(\gamma)$ by functions from $\mathcal{FC}_{0,\{e_n\}}$ along with its partial derivatives $\partial_{e_i}f$, $\partial_{e_i}^2 f$, where $i \le N$ and N is fixed.

Solutions to nonlinear equations are defined similarly. Namely, if E is a Borel set of probability densities in $L^1(\gamma)$ and $v\colon E \times X \to H$ is a Borel mapping, then a measure $\mu = \varrho\cdot\gamma$ with $\varrho \in E$ is called a solution to the nonlinear Fokker–Planck–Kolmogorov equation

$$L_{b(\varrho,\bullet)}^*\mu = 0, \quad \text{where } b(\varrho, x) = -x + v(\varrho, x), \tag{2.6}$$

if the measure μ satisfies the linear equation (2.3) with $b = b(\varrho, \bullet)$.

3. Main results

Our main result states the existence of a probability solution to the nonlinear equation with a bounded field v.

THEOREM 3.1. *Let $v\colon E \times X \to H$ be a bounded Borel mapping, where E is the subset of $L^2(\gamma)$ consisting of probability densities, equipped with the weak topology of $L^2(\gamma)$. Suppose that for each x and each $h \in H$ the function $p \mapsto (v(p,x), h)_H$ is sequentially continuous on E with respect to the weak topology of $L^2(\gamma)$. Let $b(p,x) = -x + v(p,x)$. Then equation (2.6) has a solution μ given by a density p_μ from E.*

Proof. Let us fix $p \in E$. According to [**26**], there is a unique probability density $\varrho_p \in L^1(\gamma)$ such that the measure $\varrho\cdot\gamma$ satisfies the equation $L^*_{b(p,\bullet)}(\varrho_p\cdot\gamma) = 0$ with the drift $b(p,x) = -x + v(p,x)$. It follows from [**20**] and [**4**] that $\varrho_p \in L^2(\gamma)$. Therefore, we obtain a mapping $\Psi\colon p \mapsto \varrho_p$ from E to E. We are going to apply Schauder's theorem to show that this mapping has a fixed point. Clearly, any fixed point gives a solution to the nonlinear equation.

First we observe that there is a constant C such that every probability solution to the equation $L^*_{b(p,\bullet)}(\varrho\cdot\gamma) = 0$ with $p \in E$ satisfies the estimate

$$\int_X \varrho^2\, d\gamma \le C. \tag{3.1}$$

This follows from [**4**, Theorem 2.3], which gives the bound

$$\gamma(x\colon \varrho(x) \ge t) \le e^2 \exp\bigl(-\sigma_\infty |\ln t|^2\bigr),\ t > 1,$$

where $\sigma_\infty = \bigl(2\pi\bigl\||v(p,\bullet)|_H\bigr\|_\infty\bigr)^{-2}$. If $|v(p,\bullet)|_H = 0$, then this bound means that $\varrho = 1$. Once $|2\pi v(p,x)|_H \le C_0$, we obtain

$$\gamma(x\colon \varrho(x) \ge t) \le e^2 \exp\bigl(-C_0^{-2}|\ln t|^2\bigr),\ t > 1,$$

so that

$$\int_X \varrho^2\, d\gamma \le 1 + 2e^2 \int_1^{+\infty} t \exp\bigl(-C_0^{-2}|\ln t|^2\bigr)\, dt.$$

Next, the subset S of E satisfying (3.1) is weakly compact in $L^2(\gamma)$. In addition, this subset is convex. In order to apply Schauder's theorem, it remains to verify that Ψ is continuous on S with respect to the weak topology of $L^2(\gamma)$. Note that the weak topology is metrizable on S, since S is bounded and $L^2(\gamma)$ is separable. Suppose that functions p_n converge in S to a function $p \in S$ in the weak topology. We have to show that the functions $\Psi(p_n)$ converge weakly to $\Psi(p)$. Otherwise there is a subsequence $\{p_{n_k}\}$ such that $\Psi(p_{n_k})$ converges weakly to some $g \in S$ different from $\Psi(p)$. Thus, we can assume that the whole sequence $\{\Psi(p_n)\}$ converges to $g \ne \Psi(p)$. It suffices to show that g satisfies the equation $L^*_{b(p,\bullet)}(g\cdot\gamma) = 0$, because $\Psi(p)$ is the only solution to this equation in S. Let φ be a smooth cylindrical function of the form $\varphi(x) = \psi(l_1(x),\dots,l_m(x))$, where $\psi \in C_b(\mathbb{R}^m)$ and $l_i \in X^*$. As explained above, we have

$$L_{b(p,\bullet)}\varphi = \sum_{j,k\le m} \partial_{x_k}\partial_{x_j}\psi(l_1,\dots,l_n)(j_H(l_k), j_H(l_j))_H - \sum_{j\le m} l_j(x) + \sum_{j\le m} l_j(v(p,x))$$

and similarly for p_n. Clearly,

$$\int_X \partial_{x_k}\partial_{x_j}\psi(l_1,\dots,l_n)\,\Psi(p_n)\, d\gamma \to \int_X \partial_{x_k}\partial_{x_j}\psi(l_1,\dots,l_n)\, g\, d\gamma,$$

$$\int_X l_j \Psi(p_n)\, d\gamma \to \int_X l_j g\, d\gamma$$

for all $j \le m$. In addition, $l_j(v(p_n,x)) \to l_j(v(p,x))$ and these functions are uniformly bounded. Hence

$$\int_X l_j(v(p_n,x))\Psi(p_n)(x)\,\gamma(dx) \to \int_X l_j(v(p,x))g(x)\,\gamma(dx).$$

Thus,

$$\int_X L_{b(p_n,\bullet)}\varphi\Psi(p_n)\, d\gamma \to \int_X L_{b(p,\bullet)}\varphi g\, d\gamma,$$

where the left-hand sides vanish, so the right-hand side vanishes too, which means that $L^*_{b(p,\bullet)}(g\cdot\gamma)=0$. The proof is complete. $\square$

EXAMPLE 3.2. Let $b_0\colon X\to H$ be a bounded Borel mapping and

$$b(p,x)=\int_X b_0(x-y)p(y)\,\gamma(dy).$$

Then b satisfies the hypotheses of the theorem. Indeed, if functions p_n converge in E to a function p in the weak topology of $L^2(\gamma)$, then, for every x, we have

$$\int_X b_0(x-y)p_n(y)\,\gamma(dy)\to\int_X b_0(x-y)p(y)\,\gamma(dy).$$

Due to the boundedness of b_0 we have convergence $b(p_n,x)\to b(p,x)$ in H. In particular, we have $(b(p_n,x),h)_H\to(b(p,x),h)_H$ for every $h\in H$. For every fixed p the mapping $x\mapsto b(p,x)$ on X is Borel measurable. For every fixed x the mapping $p\mapsto b(p,x)$ is continuous on E in the weak topology. Since the sets

$$E_N=\{f\in E\colon \|f\|_{L^2(\gamma)}\le N\}$$

are compact metrizable in the weak topology, we conclude that b is jointly Borel measurable on each $E_N\times X$ (see [**3**, Exercise 6.10.40]). Then b is jointly Borel measurable on all of $E\times X$.

The theorem proved above can be extended to more general unbounded fields v and non-constant diffusion operators. However, this requires more technicalities based on the results from [**5**], [**9**], [**4**], and [**20**] and will be considered separately.

We now assume that v depends additionally on a parameter u from a complete separable metric space U and

$$(p,x,u)\mapsto v_u(p,x),\ \ E\times X\times U\to H$$

is a bounded Borel mapping.

PROPOSITION 3.3. *There are solutions $p_u\in E$ to the equations $L^*_{b_u(p_u,\bullet)}(p_u\cdot\gamma)=0$ such that the mapping $u\mapsto p_u$ from U to E is Borel measurable.*

PROOF. Let S be the weakly compact set in E introduced in the proof of the theorem above. For every $u\in U$ the set S_u of all solutions $p\in S$ to the nonlinear equation $L^*_{b_u(p,\bullet)}(p\cdot\gamma)=0$ is compact. Let us show that the

$$W=\{(u,p)\colon u\in U,\,p\in S_u\}$$

is Borel in $U\times E$. Once this is done, we can apply the classical measurable selection theorem, which gives a Borel mapping $F\colon U\to S$ such that $F(u)\in S_u$ for every $u\in U$ (see [**21**, §35]).

As explained above, we can assume that there is a countable family of functionals $l_n\in X^*$ separating points in X, so we can assume that the functionals $l_n=\widehat{e}_n$ play this role (or simply deal with $\mathbb{R}^\infty$ and take the coordinate functions). Then equation $L_b^*(p\cdot\gamma)$ with respect to $\mathcal{FC}$ is equivalent to this equation with respect to $\mathcal{FC}_{0,\{e_n\}}$ (see the discussion in Section 2). In turn, testing the latter reduces to checking (2.4) for a suitable countable family $\{\varphi_j\}\subset\mathcal{FC}_{0,\{e_n\}}$. Therefore, the equality $L^*_{b_u(p,\bullet)}(p\cdot\gamma)=0$ is equivalent to the countable system of relations

$$\int_X\sum_{i=1}^N[\partial_{e_i}^2\varphi_j-l_i\partial_{e_i}\varphi_j+(v_u(p,\bullet),e_i)_H]p\,d\gamma=0.$$

Thus, it remains to observe that this integral defines a Borel function on $U \times S$. This is obvious for the terms with partial derivatives. Let us consider the function

$$\int_X (v_u(p, \bullet), e_i)_H] p\, d\gamma.$$

Using an orthonormal basis $\{\psi_j\}$ in $L^2(\gamma)$, we obtain the functions

$$\int_X p\psi_j\, d\gamma \int_X (v_u(p, \bullet), e_i)_H] \psi_j\, d\gamma,$$

where the first integral is continuous in p and the second one is Borel measurable in u, because the function $(u, x) \mapsto (v_u(p, x), e_i)_H]\psi_j(x)$ is jointly measurable. □

Finally, we consider a more general situation where v still takes values in H, but is not assumed uniformly bounded with respect to the norm of H. We now assume instead that

$$|(v(\mu, x), e_n)_H| \le C$$

for some constant C and some orthonormal basis $\{e_n\}$ in H. To simplify our presentation we assume that $X = \mathbb{R}^\infty$ and $H = l^2$ with its standard basis $\{e_n\}$, however, a more abstract formulation can be derived easily from what follows. Now no Gaussian measure is fixed, so the mapping v is defined on $\mathcal{P}(X) \times X$, where $\mathcal{P}(X)$ is the space of all Borel probability measures on X equipped with the weak topology (it is well known that this topology is metrizable by a complete separable metric, for example, by the Prohorov metric or by the Kantorovich–Rubinshtein metric).

THEOREM 3.4. *Suppose that the functions $(\mu, x) \mapsto v_n(\mu, x) := (v(\mu, x), e_n)_H$ are continuous on $\mathcal{P}(X) \times X$ and uniformly bounded. Then there is a probability solution to equation* (2.6) *with respect to the class $\mathcal{FC}_{0,\{e_n\}}$.*

PROOF. Let us take any sequence of positive numbers α_n with $T := \sum_{n=1}^\infty \alpha_n < \infty$. We construct our solution on the weighted Hilbert space $E \subset X$ of sequences with finite norm

$$|x|_E = \Bigl(\sum_{n=1}^\infty \alpha_n x_n^2\Bigr)^{1/2}.$$

For every $k \in \mathbb{N}$ we define a k-dimensional mapping $v^k\colon \mathcal{P}(\mathbb{R}^k) \times \mathbb{R}^k \to \mathbb{R}^k$ by setting $v^k_n(\mu, x) = v_n(\mu, x)$, $n \le k$, where each vector $(x_1, \ldots, x_k) \in \mathbb{R}^k$ is identified with the vector $(x_1, \ldots, x_k, 0, 0, \ldots) \in \mathbb{R}^\infty$ and $\mathcal{P}(\mathbb{R}^k)$ is naturally identified with the subset of $\mathcal{P}(X)$ consisting of measures concentrated on $\mathbb{R}^k$.

It follows from our first theorem that for each k there is a probability solution $\mu_k \in \mathcal{P}(\mathbb{R}^k)$ to the nonlinear equation with the drift v^k, moreover, we have that such a solution is absolutely continuous with respect to the standard Gaussian measure γ_k on $\mathbb{R}^k$. Indeed, we can consider $v^k_n(\mu, x)$ only for absolutely continuous measures and these functions satisfy the continuity assumption from that theorem, because weak convergence of densities in $L^2(\gamma_k)$ implies weak convergence of the corresponding measures.

Let us take $V(x) = |x|_E^2 = \sum_{n=1}^\infty \alpha_n x_n^2$ as a Lyapunov function for our finite-dimensional equations. We have

$$\begin{aligned} L_{v^k(\mu,\bullet)} V(x) &= 2\sum_{n=1}^k \alpha_n - 2\sum_{n=1}^k \alpha_n x_n^2 + 2\sum_{n=1}^k \alpha_n x_n v^k_n(\mu, x) \\ &\le 2T - 2V(x) + 2CV(x)^{1/2}T^{1/2} \le 2T + C^2T - V(x). \end{aligned}$$

Since μ_k satisfies a linear equation with an operator satisfying the above estimate with V, it follows from standard a priori estimates for solutions to linear Fokker–Planck–Kolmogorov equations (see [**6**]) that

$$\int_{\mathbb{R}^k} V\, d\mu_k \le (2+C^2)T.$$

The sets $\{V \le R\}$ are compact in $\mathbb{R}^\infty$, hence it follows from the Chebyshev inequality that the measures μ_k are uniformly tight on $\mathbb{R}^\infty$. Passing to a subsequence, we can assume that they converge weakly to some probability measure μ on $\mathbb{R}^\infty$.

Let us verify that μ is a solution (2.6). Let $\varphi \in C_0^\infty(\mathbb{R}^k)$. We can assume that $|\partial_{x_j}\varphi| \le 1$. For all $n \ge k$ we have

$$\int_{\mathbb{R}^n} [\Delta\varphi(x) - (x, \nabla\varphi(x)) + (v(\mu_n, x), \nabla\varphi(x))]\, \mu_n(dx) = 0.$$

Obviously,

$$\int_{\mathbb{R}^n} [\Delta\varphi(x) - (x, \nabla\varphi(x))]\, \mu_n(dx) \to \int_{\mathbb{R}^\infty} [\Delta\varphi(x) - (x, \nabla\varphi(x))]\, \mu(dx)$$

as $n \to \infty$. Let us show that for each $j \le k$ we have

$$\int_{\mathbb{R}^n} v_j(\mu_n, x)\partial_{x_j}\varphi(x)\, \mu_n(dx) \to \int_{\mathbb{R}^\infty} v_j(\mu, x)\partial_{x_j}\varphi(x)\, \mu(dx).$$

Let $\varepsilon > 0$. We have $|v_j(\mu_n, x)\partial_{x_j}\varphi(x)| \le C$. There is $R > 0$ such that

$$\mu_n(V > R) + \mu(V > R) \le \varepsilon C^{-1}$$

for all n. Hence the integrals of $v_j(\mu_n, x)\partial_{x_j}\varphi(x)$ over the set $\{V > R\}$ do not exceed ε. The sequence $\{\mu_n\}$ with the added limit μ is a compact set. By the uniform continuity of v_j on compacts sets we conclude that $v_j(\mu_n, x) \to v_j(\mu, x)$ uniformly on $\{V \le R\}$. Hence for all n large enough we have

$$\left| \int_{\{V\le R\}} v_j(\mu_n, x)\partial_{x_j}\varphi(x)\, \mu_n(dx) - \int_{\{V\le R\}} v_j(\mu, x)\partial_{x_j}\varphi(x)\, \mu_n(dx) \right| < \varepsilon.$$

In addition, for all n large enough

$$\left| \int_X v_j(\mu, x)\partial_{x_j}\varphi(x)\, \mu_n(dx) - \int_X v_j(\mu, x)\partial_{x_j}\varphi(x)\, \mu(dx) \right| < \varepsilon.$$

For such n we obtain

$$\left| \int_{\{V\le R\}} v_j(\mu, x)\partial_{x_j}\varphi(x)\, \mu_n(dx) - \int_{\{V\le R\}} v_j(\mu, x)\partial_{x_j}\varphi(x)\, \mu(dx) \right| < 2\varepsilon,$$

hence

$$\left| \int_{\{V\le R\}} v_j(\mu_n, x)\partial_{x_j}\varphi(x)\, \mu_n(dx) - \int_{\{V\le R\}} v_j(\mu, x)\partial_{x_j}\varphi(x)\, \mu(dx) \right| < 3\varepsilon.$$

Therefore, we have the desired convergence of the integrals of $v_j(\mu_n, x)\partial_{x_j}\varphi(x)$, which shows that μ is a solution to our equation. □

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References

[1] Albeverio S., Bogachev V., Röckner M. On uniqueness of invariant measures for finite- and infinite-dimensional diffusions. Comm. Pure Appl. Math. 52 (1999), N 3, 325–362.

[2] Bogachev V.I. Gaussian measures. Amer. Math. Soc., Providence, Rhode Island, 1998.

[3] Bogachev V.I. Measure theory. V. 1, 2. Springer, New York, 2007.

[4] Bogachev V.I., Kosov E.D., Shaposhnikov A.V. Regularity of solutions to Kolmogorov equations with perturbed drifts. Potential Anal. 58 (2023), N 4, 681–702.

[5] V. I. Bogachev, N. V. Krylov, M. Röckner, Regularity of invariant measures: the case of non-constant diffusion part, J. Funct. Anal. 138 (1996), 223-–242.

[6] V.I. Bogachev, N.V. Krylov, M. Röckner, S.V. Shaposhnikov, Fokker–Planck–Kolmogorov equations, Amer. Math. Soc., Providence, Rhode Island, 2015.

[7] V.I. Bogachev, M. Röckner, Regularity of invariant measures on finite and infinite dimensional spaces and applications, J. Funct. Anal. 133 (1995), 168–223.

[8] V.I. Bogachev, M. Röckner, T.S. Zhang, Existence and uniqueness of invariant measures: an approach via sectorial forms, Appl. Math., Optim. 41 (2000), 87–109.

[9] Bogachev V.I., Shaposhnikov A.V., Shaposhnikov S.V. Log-Sobolev-type inequalities for solutions to stationary Fokker–Planck–Kolmogorov equations. Calc. Var. Partial Differ. Equ. 58 (2019), N 5, Paper No. 176, 16 pp.

[10] V.I. Bogachev, S.V. Shaposhnikov, Nonlinear Fokker–Planck–Kolmogorov equations, Uspehi Mat. Nauk 79:5 (2024), 3–60 (in Russian); English transl.: Russian Math. Surveys 79:5 (2024).

[11] Cerrai S. Second order PDE's in finite and infinite dimension. A probabilistic approach. Lecture Notes in Mathematics, 1762. Springer-Verlag, Berlin, 2001.

[12] Da Prato G. Kolmogorov equations for stochastic PDEs. CRM Barcelona. Birkhäuser, Basel, 2004.

[13] Da Prato G., Zabczyk J. Second order partial differential equations in Hilbert spaces. Cambridge University Press, Cambridge, 2002.

[14] Dalecky Yu.L., Fomin S.V. Measures and differential equations in infinite-dimensional space. Transl. from the Russian. With additional material by V.R. Steblovskaya, Yu.V. Bogdansky and N.Yu. Goncharuk. Kluwer, Dordrecht, 1991.

[15] Gordina M., Röckner M., Teplyaev A. Ornstein-Uhlenbeck processes with singular drifts: integral estimates and Girsanov densities. Probab. Theory Related Fields 178 (2020), no. 3-4, 861–891.

[16] Gross L. Integration and nonlinear transformations in Hilbert space. Trans. Amer. Math. Soc. 1960. V. 94, N3. P. 404–440.

[17] Gross L. Harmonic analysis on Hilbert space. Mem. Amer. Math. Soc. 1963. V. 46. P. 1–62.

[18] Gross L. Potential theory on Hilbert space. J. Funct. Anal. 1967. V. 1, N2. P. 123–181.

[19] Gross L. Logarithmic Sobolev inequalities. Amer. J. Math. 1975. V. 97, N4. P. 1061–1083.

[20] Hino M. Existence of invariant measures for diffusion processes on a Wiener space. Osaka J. Math. 35 (1998), no. 3, 717–734.

[21] Kechris A.S. Classical descriptive set theory. Springer, Berlin – New York, 1995.

[22] Kuo H.H. Gaussian measures in Banach spaces. Lecture Notes in Mathematics, Vol. 463. Springer-Verlag, Berlin – New York, 1975.

[23] Kuo H.H., Piech M.A. Stochastic integrals and parabolic equations in abstract Wiener space. Bull. Amer. Math. Soc. 79 (1973), 478–482.

[24] Piech M.A. Parabolic equations associated with the number operator. Trans. Amer. Math. Soc. 1974. V. 194. P. 213–222.

[25] Piech M.A. The Ornstein,–Uhlenbeck semigroup in an infinite dimensional L^2 setting. J. Funct. Anal. 1975. V. 18, N3. P. 271–285.

[26] Shigekawa I. Existence of invariant measures of diffusions on an abstract Wiener space. Osaka J. Math. 1987. V. 24, N1. P. 37–59.

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