

Angular maps for visualizing rotational dynamics

Wolf-Jürgen Beyn* Thorsten Hüls*

Abstract. We develop and analyze angular maps as a numerical tool for visualizing and detecting rotational dynamics in nonlinear, nonautonomous dynamical systems in discrete and continuous time. An angular map assigns to each initial point the angular spectrum of the variational equation along the corresponding trajectory. This spectrum describes the long-time average rotation of subspaces transported by the linearized dynamics, measured by maximal principal angles. Building on the theory of angular spectra, we develop efficient numerical algorithms based on forward and backward subspace iteration. We prove that these algorithms asymptotically provide angular spectral values for subspaces that are dominant in either forward or backward time. Applications to Hénon maps, a planar flow, and the Lorenz system illustrate how angular maps reveal rotational features across phase space. For continuous-time systems, we prove that suitably rescaled angular spectra of exact time-step maps converge to the continuous angular spectrum in the Hausdorff metric as the step size tends to zero. For autonomous systems, we also justify a simplified algorithm that uses successive trajectory points to approximate the angular range associated with the flow direction.

Key words. Angular maps, angular spectrum, principal angles, nonautonomous dynamical systems, Sacker-Sell spectrum, numerical approximation.

AMS subject classifications. 37C60, 37M25, 34D09, 65Q10.

1. Introduction. The main goal of this paper is to quantify the rotational behavior of dynamical systems in discrete and continuous time. We first consider discrete-time systems

$$(1.1) \quad x_{n+1} = F_n(x_n), \quad n \in \mathbb{N}_0, \quad x_0 \in \mathbb{R}^d,$$

where $F_n \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$. With each trajectory $(x_n)_{n \in \mathbb{N}_0}$ we associate the variational equation

$$(1.2) \quad u_{n+1} = A_n u_n, \quad n \in \mathbb{N}_0, \quad \text{with } A_n = DF_n(x_n) \text{ invertible}$$

and ask for its angular spectrum. In [5] we introduced the notion of an angular spectrum for general linear nonautonomous dynamical systems. For every fixed $s < d$ the angular spectrum of dimension s measures the long-time rotational behavior of all s -dimensional subspaces as they evolve under the linear system (1.2); see Definition 1.1 below.

Our aim is to visualize how the long-time rotational behavior varies across phase space. Think of leaves floating in a current, as illustrated in Figure 1.1. We quantify the rotation that these leaves experience on average along their trajectories by considering the angular spectra associated with a multitude of initial values.

We make the dependence on the initial value explicit by writing

$$(1.3) \quad x_n = x_n(x_0), \quad A_n = A_n(x_0) = DF_n(x_n(x_0)), \quad n \in \mathbb{N}_0.$$

Correspondingly, we let $\Phi(n, m; x_0)$ denote the solution operator of (1.2), defined by $\Phi(n, m; x_0) = A_{n-1}(x_0) \cdot \dots \cdot A_m(x_0)$ for $n > m$, $\Phi(n, n; x_0) = I_d$, and $\Phi(n, m; x_0) = A_n^{-1}(x_0) \cdot \dots \cdot A_{m-1}^{-1}(x_0)$ for $n < m$.

*Department of Mathematics, Bielefeld University, 33501 Bielefeld, Germany
beyn@math.uni-bielefeld.de, huels@math.uni-bielefeld.de

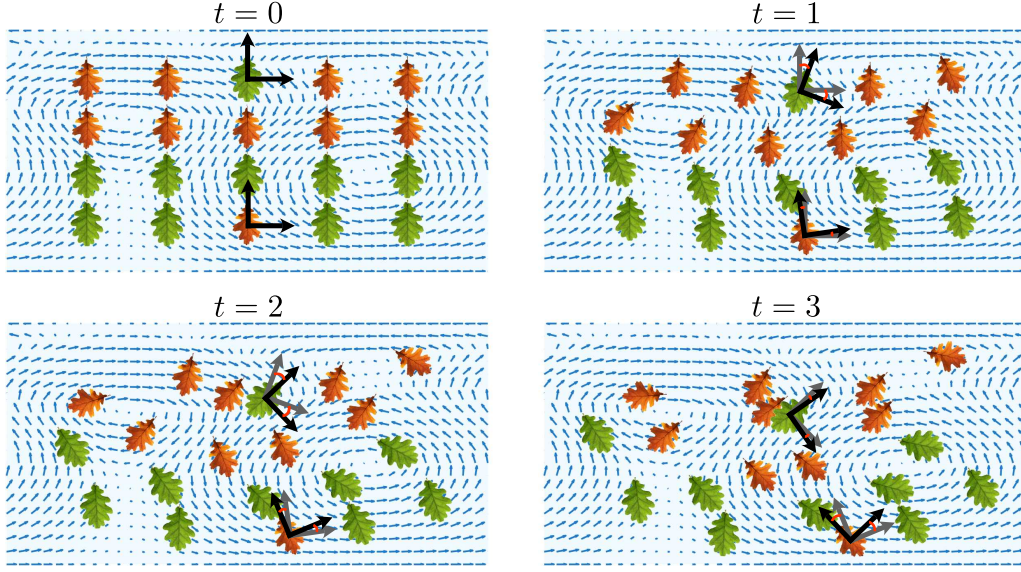


Figure 1.1: Leaves floating in a vector field. For two selected leaves, orthogonal coordinate frames indicate their orientations at the current time (black) and at the previous time (gray). Red arcs indicate the rotation angles.

Our objects of study live in the Grassmannian

$$\mathcal{G}(s, d) = \{V \subseteq \mathbb{R}^d : V \text{ is a subspace of dimension } s\},$$

which is a smooth manifold of dimension $s(d - s)$ and a metric space w.r.t. the largest principal angle $\angle(V, W) \in [0, \frac{\pi}{2}]$ between two elements $V, W \in \mathcal{G}(s, d)$. We recall this notion in Section 2 and summarize some of its basic properties.

The solution operator of (1.2), (1.3) is then used to form the average of principal angles between iterates of an element $V \in \mathcal{G}(s, d)$

$$(1.4) \quad \begin{aligned} \mathbb{N} \times \mathcal{G}(s, d) \times \mathbb{R}^d &\rightarrow [0, \frac{\pi}{2}] \\ \alpha_s^{\text{disc}}(n, V; x_0) &\mapsto \frac{1}{n} \sum_{j=1}^n \angle(\Phi(j-1, 0; x_0)V, \Phi(j, 0; x_0)V). \end{aligned}$$

These are the key quantities which determine the angular spectrum.

Definition 1.1. For $V \in \mathcal{G}(s, d)$ and $x_0 \in \mathbb{R}^d$ the interval

$$(1.5) \quad I_s^{\text{disc}}(V; x_0) = \left[\liminf_{n \rightarrow \infty} \alpha_s^{\text{disc}}(n, V; x_0), \overline{\lim}_{n \rightarrow \infty} \alpha_s^{\text{disc}}(n, V; x_0) \right]$$

is called the s -dimensional **angular range** of V for the system (1.2). The s -dimensional **angular spectrum** is defined by

$$\Sigma_s^{\text{disc}}(x_0) = \text{cl} \left[\bigcup_{V \in \mathcal{G}(s, d)} I_s^{\text{disc}}(V; x_0) \right].$$

The map that assigns to any x_0 in some domain $\mathcal{B} \subseteq \mathbb{R}^d$ the closed set $\Sigma_s^{\text{disc}}(x_0)$ is called the **angular map** of (1.2) in $\mathcal{B}$.

Note that Σ_s^{disc} is called *outer* angular spectrum in [5]. We omit the term *outer* in this paper, since other variants of the angular spectrum (see [5, Section 5.3]) are more difficult to access theoretically and extremely costly to compute numerically. The superscript “disc” is used to distinguish the angular spectrum in discrete time from its continuous time counterpart.

The continuous time angular spectrum Σ_s^{cont} is discussed in Section 4. According to [4] it is based on the following quantities, defined for $T > 0$ and $V \in \mathcal{G}(s, d)$:

$$(1.6) \quad \alpha_s^{\text{cont}}(T, V; x_0) = \frac{1}{T} \int_0^T \|(I_d - P_{\Phi(t,0;x_0)V})A(t; x_0)P_{\Phi(t,0;x_0)V}\| dt.$$

Here $\Phi(t, s; x_0)$ is the solution operator of a linear continuous time system

$$\dot{u} = A(t; x_0)u, \quad u(s) = u_s,$$

which depends on the initial value x_0 of a nonlinear system; see (4.1), (4.2) below. Furthermore, the symbol P_V in (1.6) denotes the orthogonal projector onto the subspace $V \in \mathcal{G}(s, d)$, and $\|\cdot\|$ is the spectral norm. The integrand measures the instantaneous rate of rotation of the evolving subspace. The continuous angular spectrum Σ_s^{cont} is then defined in an analogous fashion by considering the limit $T \rightarrow \infty$; see Definition 4.1.

Our planar flow example illustrates how angular maps reveal differences in the mean rotation of infinitesimal perturbations that are not apparent from the mean rotation along the flow direction alone. In particular, the latter may lie strictly between the minimal and maximal angular values; see Figure 4.6.

We develop theoretical and numerical foundations of angular maps through the following contributions:

- We extend the algorithm from [5] to compute angular maps. By color-coding the minimal and maximal spectral values within the region of interest, we obtain a global map of the rotational dynamics.
- We develop efficient algorithms based on forward and backward iteration of subspaces. These techniques apply to several examples and avoid the costly computation of the dichotomy spectrum and the corresponding spectral bundles. For generic initial subspaces, we establish conditions under which the angular averages computed by forward iteration asymptotically agree with those associated with the fastest spectral subspaces.
- We establish a uniform error estimate between the continuous angular averages $\alpha_s^{\text{cont}}(T, V; x_0)$ and their suitably scaled discrete counterparts $\alpha_s^{\text{step}}(h, T, V; x_0)$, obtained from exact flow values at time steps of length h . This estimate yields convergence of the corresponding angular spectra to Σ_s^{cont} in the Hausdorff metric as $h \rightarrow 0$.

The remainder of the paper is organized as follows.

In Section 2 we introduce numerical algorithms based on forward and backward subspace iteration and illustrate them using various versions of the Hénon map.

In Section 3 we relate these algorithms to the reduction theory developed in [3]. This theory uses the exponential dichotomy spectrum of the variational equation and its associ-

ated spectral fibers to reduce the computation of angular spectra to a distinguished family of subspaces, the so-called trace spaces. It allows us to identify which parts of the angular spectrum are captured by the algorithms of Section 2. We then show how forward and backward iteration can be combined to recover further spectral fibers and their associated angular values. The reduction to trace spaces also provides the basis for a general algorithm for computing angular maps, including approaches for determining the smallest and largest angular values. We illustrate these ideas further using the area-preserving Hénon map.

In Section 4 we extend our approach to continuous-time systems and illustrate it for a planar flow and the three-dimensional Lorenz system. We establish the approximation and convergence results stated above (Theorems 4.3 and 4.6). Finally, we present and analyze a simplified algorithm for autonomous systems that uses successive trajectory points to approximate the angular range associated with the flow direction.

2. Principal angles and algorithms for their numerical approximation. The maximal principal angle of two subspaces $V, W \in \mathcal{G}(s, d)$ is denoted by $\angle(V, W)$. It can be characterized as follows (see [2, Prop. 2.3])

$$\angle(V, W) = \max_{\substack{v \in V \\ v \neq 0}} \min_{\substack{w \in W \\ w \neq 0}} \angle(v, w) = \arccos \left(\min_{\substack{v \in V \\ \|v\|=1}} \max_{\substack{w \in W \\ \|w\|=1}} v^\top w \right).$$

Note that $\angle(v, w) = \angle(\text{span}(v), \text{span}(w)) = \arccos \left(\frac{|v^\top w|}{\|v\| \|w\|} \right) \in [0, \frac{\pi}{2}]$ measures the principal angle between the subspaces $\text{span}(v)$ and $\text{span}(w)$. It ignores the orientation of their spanning vectors v and w measured by $\angle(v, w) = \arccos \left(\frac{v^\top w}{\|v\| \|w\|} \right) \in [0, \pi]$. For general subspaces $V, W \in \mathcal{G}(s, d)$, let the columns of the $d \times s$ -matrices V_B and W_B form an orthonormal basis of V resp. W ; then the maximal principal angle satisfies $\angle(V, W) = \arccos(\sigma_s)$, where $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_s \geq 0$ denote the singular values of $V_B^\top W_B$; see [7, Ch. 6.4.3]. The maximal principal angle defines a metric on $\mathcal{G}(s, d)$ and so does the expression

$$(2.1) \quad \|P_V - P_W\| = \sin(\angle(V, W)), \quad V, W \in \mathcal{G}(s, d),$$

where $\|\cdot\|$ denotes the spectral norm and P_V, P_W are the orthogonal projectors onto the subspaces V and W , respectively [7, Ch. 6.4.3], [4, Prop. 2.3]. A further equation relates the angle $\angle(V, W)$ to the distance of the unit ball in one subspace from the other subspace; see [12, Ch.II.4]:

$$(2.2) \quad \sup\{\text{dist}(v, W) : v \in V, \|v\| = 1\} = \sin(\angle(V, W)).$$

Similarly, one finds for the minimal angle $\theta_{\min}(V, W) = \arccos(\sigma_1)$ between V and W

$$(2.3) \quad \inf\{\text{dist}(v, W) : v \in V, \|v\| = 1\} = \sin(\theta_{\min}(V, W)).$$

Sometimes, it is useful to work with orthogonal complements of subspaces rather than the subspaces themselves (see Algorithm 3), using the following Lemma.

Lemma 2.1. *For $V, W \in \mathcal{G}(s, d)$ and nonsingular $A \in \mathbb{R}^{d,d}$ the following relations hold:*

$$(2.4) \quad \angle(V, W) = \angle(V^\perp, W^\perp),$$

$$\angle(V, AV) = \angle(V^\perp, A^{-\top} V^\perp).$$

Proof. The equality (2.4) follows from (2.1) and the fact that $I - P_V = P_{V^\perp}$ is the projector onto the orthogonal complement:

$$\sin(\angle(V, W)) = \|P_V - P_W\| = \|(I_d - P_V) - (I_d - P_W)\| = \sin(\angle(V^\perp, W^\perp)).$$

Further, the algebraic identity $(AV)^\perp = A^{-\top}V^\perp$ and (2.4) yield

$$\angle(V, AV) = \angle(V^\perp, (AV)^\perp) = \angle(V^\perp, A^{-\top}V^\perp).$$

■

2.1. One-dimensional spectra in discrete time – the fast subspace. For the discrete time system (1.1), we propose a simple algorithm to illustrate the rotational dynamics of several trajectories. The area of interest is given for dimension $d = 2$ resp. $d = 3$ by

$$\mathcal{B}^2 = [a_-, a_+] \times [b_-, b_+] \quad \text{and} \quad \mathcal{B}^3 = [a_-, a_+] \times [b_-, b_+] \times [c_-, c_+].$$

Let $L \in \mathbb{N}$ denote the resolution of the grid and define

$$h_a = \frac{a_+ - a_-}{L}, \quad h_b = \frac{b_+ - b_-}{L} \quad \text{and in the case } d = 3 \quad h_c = \frac{c_+ - c_-}{L}.$$

For $d = 2$ we define the (i, j) -th box for $i, j \in \{1, \dots, L\}$ as

$$\mathcal{B}_{i,j} = [a_- + (i-1)h_a, a_- + ih_a] \times [b_- + (j-1)h_b, b_- + jh_b].$$

In Algorithm 1 we let $N \in \mathbb{N}$ denotes the trajectory length and assume that the Jacobians DF_{n-1} are available. The algorithm aims to compute the average rotation angle w.r.t. the fastest direction.

Algorithm 1 Average rotation angle w.r.t. the fastest direction ($s = 1$)

```

1: for  $i = 1, \dots, L$  do
2:   for  $j = 1, \dots, L$  do
3:     for  $k = 1, \dots, L$  (if  $d = 3$ ) do
4:       Choose the midpoint  $x_0 \in \mathcal{B}_{i,j,(k)}$ 
5:       Choose  $V_0 \in \mathbb{R}^d$ ,  $\|V_0\| = 1$  at random
6:       for  $n = 1, \dots, N$  do
7:          $x_n = F_{n-1}(x_{n-1})$ 
8:          $Y = DF_{n-1}(x_{n-1})V_{n-1}$ 
9:          $V_n = \frac{Y}{\|Y\|}$ 
10:       $\Gamma_{i,j,(k)} = \frac{1}{N} \sum_{\ell=1}^N \angle(V_{\ell-1}, V_\ell)$ 
11:      if  $x_n \in \mathcal{B} \forall n \in \{1, \dots, N\}$  then
12:        Color the box  $\mathcal{B}_{i,j,(k)}$  with the color that corresponds to the angle  $\Gamma_{i,j,(k)}$ 

```

In Algorithm 1 we include the case $d = 3$ and define the (i, j, k) -th box for $i, j, k \in \{1, \dots, L\}$ as

$$\mathcal{B}_{i,j,k} = [a_- + (i-1)h_a, a_- + ih_a] \times [b_- + (j-1)h_b, b_- + jh_b] \times [c_- + (k-1)h_c, c_- + kh_c].$$

We apply Algorithm 1 to the following two- and three-dimensional Hénon systems:

$$(2.5) \quad f_2 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 + x_2 - 1.4x_1^2 \\ 0.3x_1 \end{pmatrix} \quad \text{and} \quad f_3 : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 + x_3 - 1.4x_1^2 \\ x_1 + x_3 \\ 0.2x_1 + 0.1x_2 \end{pmatrix}.$$

For the two-dimensional Hénon map, we consider the box $\mathcal{B} = [-1.5, 1.5]^2$ with resolution $L = 10^3$. We use $N = 30$ iterates for the left diagram in Figure 2.1 and $N = 10^4$ for the right diagram. In addition, we plot the Hénon attractor in black to indicate its position within the domain of colored points that do not leave the prescribed box.

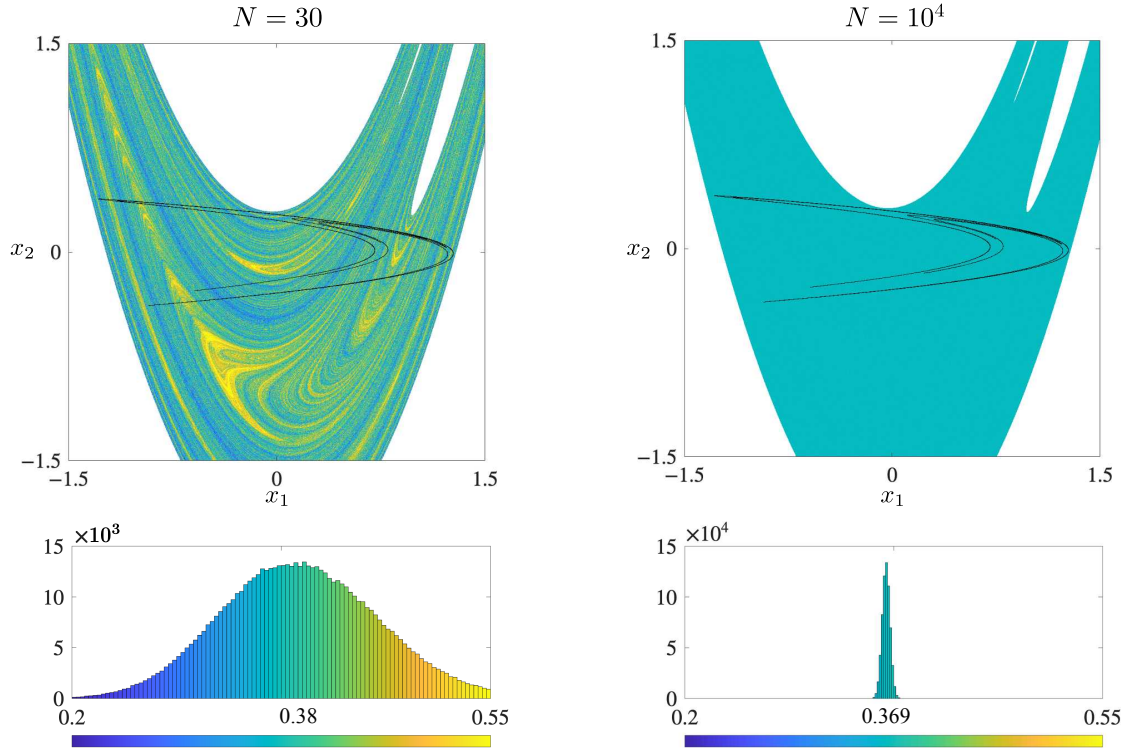


Figure 2.1: Algorithm 1 applied to f_2 (see (2.5)) with $N = 30$ (left) and $N = 10^4$ (right).

In the second row of the figure we provide a histogram of approximate angular spectral data for those initial values among the $L^2 = 10^6$ grid points whose trajectories remain in $\mathcal{B}$. The color coding of points in phase space is synchronized with their corresponding position in the histogram. We observe a marked difference between the results for short ($N = 30$) and long trajectories ($N = 10^4$). For long trajectories, angular spectral values show a very narrow distribution and almost no dependence on the initial points in the domain of attraction. However, for short trajectories, we observe a broad distribution function with high values (yellow) and low values (blue) originating from different parts of the domain of attraction.

For the three-dimensional Hénon map, we choose the box $\mathcal{B} = [-2, 2] \times [-3, 3] \times [-3, 3]$ with resolution $L = 300$. We use $N = 30$ steps for the left diagrams in Figure 2.2 and

$N = 10^4$ for the right diagrams. These diagrams look very similar to the $2D$ case: a broad distribution around a specific angular value related to a structured domain of attraction and a very narrow distribution centered at $\alpha_1^{\text{disc}} = 0.836$ almost uniformly for all initial values in the domain of attraction. In Section 3 below we show where this angular spectral value is located in the angular spectrum.

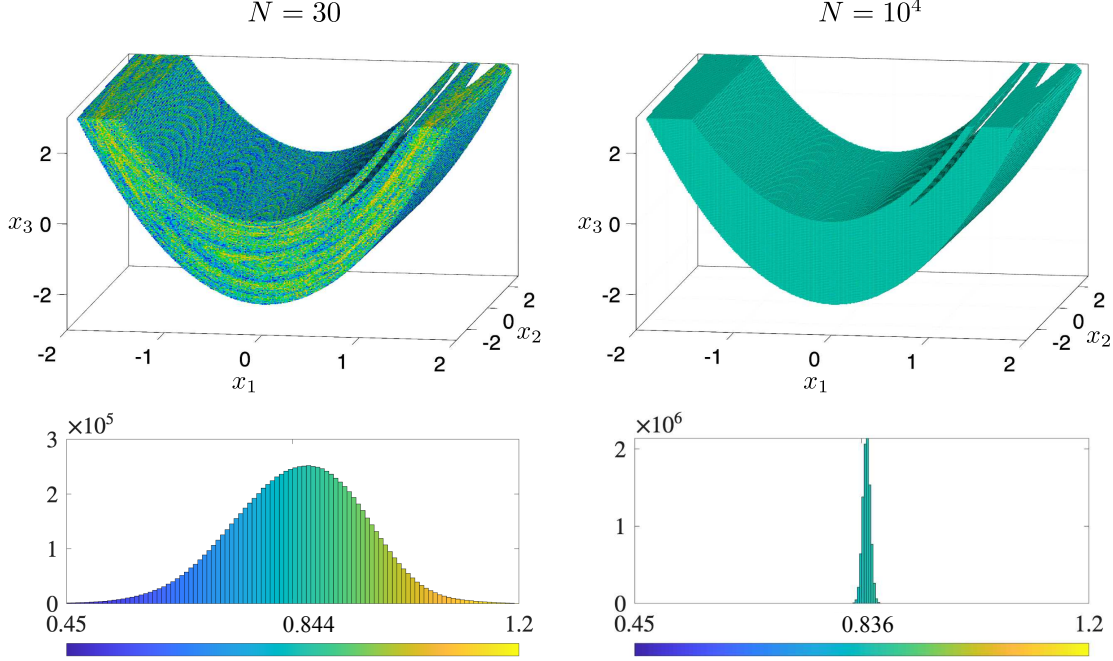


Figure 2.2: Algorithm 1 applied to f_3 (see (2.5)) with $N = 30$ (left) and $N = 10^4$ (right).

2.2. A QR -like algorithm for multidimensional angular maps. We generalize Algorithm 1 to the general case $s \geq 1$. For this step we use QR factorization analogous to the transition from the power method to subspace iteration for eigenvalue problems; see [7, Ch. 7.3, 7.6]. In line 9 we use the thin QR factorization for which $V_n \in \mathbb{R}^{d,s}$, $V_n^\top V_n = I_s$

Algorithm 2 QR -like algorithm replacing lines 5–10 of Algorithm 1 ($s \geq 1$)

```

5: Choose  $V_0 \in \mathbb{R}^{d,s}$ ,  $V_0^\top V_0 = I_s$  at random
6: for  $n = 1, \dots, N$  do
7:    $x_n = F_{n-1}(x_{n-1})$ 
8:    $Y = DF_{n-1}(x_{n-1})V_{n-1}$ 
9:    $(V_n, R_n) = QR(Y)$ 
10:  $\Gamma_{i,j,(k)} = \frac{1}{N} \sum_{\ell=1}^N \angle(V_{\ell-1}, V_\ell)$ 

```

and $R_n \in \mathbb{R}^{s,s}$ is upper triangular; see [7, Theorem 5.2.3]. For large N , the value $\Gamma_{i,j,(k)}$ approximates some value in the angular range $I_s^{\text{disc}}(V_0; x_0)$. In Section 3 we will discuss how the specific value selected by the algorithm relates to the invariant foliation induced

by the dichotomy spectrum of the linearized equation (1.2).

We apply Algorithm 2 to the three-dimensional Hénon system f_3 (see (2.5)) in case $s = 2$, using the setup from Figure 2.2. We note that computing the QR factorizations is considerably more time-consuming than the normalization performed in line 9 of Algorithm 1. We present the histograms for $N = 30$ and $N = 10^4$ in Figure 2.3, which provide the most relevant information.

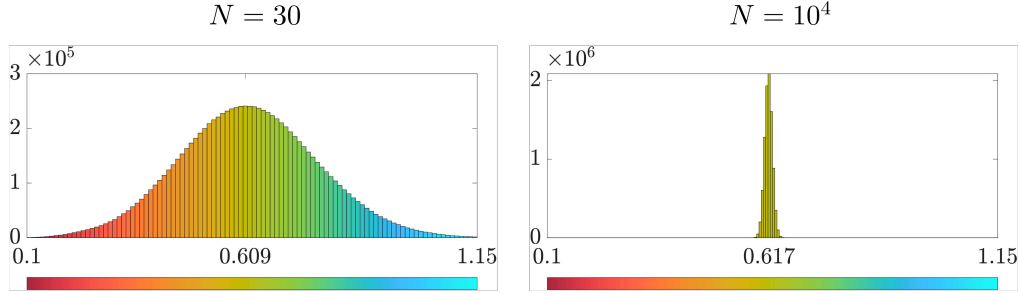


Figure 2.3: Algorithm 2 applied to f_3 (see (2.5)) for $s = 2$ with $N = 30$ (left) and $N = 10^4$ (right).

The orthogonal complements of elements $V \in \mathcal{G}(s, d)$ satisfy $V^\perp \in \mathcal{G}(d - s, d)$. Therefore, if $s > \frac{d}{2}$, it is more efficient to iterate orthogonal complements of smaller dimension and employ Lemma 2.1 for computing angles. This leads to the following algorithm:

Algorithm 3 *QR-like algorithm for orthogonal complements*,
replacing line 5–10 of Algorithm 1 ($s \geq 1$)

- 5: Choose $U_0 \in \mathbb{R}^{d, d-s}$, $U_0^\top U_0 = I_{d-s}$ at random
 - 6: **for** $n = 1, \dots, N$ **do**
 - 7: $x_n = F_{n-1}(x_{n-1})$
 - 8: Solve $DF_{n-1}(x_{n-1})^\top Z = U_{n-1}$
 - 9: $(U_n, S_n) = QR(Z)$
 - 10: $\Gamma_{i,j,(k)} = \frac{1}{N} \sum_{\ell=1}^N \angle(U_{\ell-1}, U_\ell)$
-

If we choose $U_0 \in \mathbb{R}^{d, d-s}$ in Step 5 such that $U_0^\top V_0 = 0$ and $U_0^\top U_0 = I_{d-s}$ holds for the initial space V_0 from Algorithm 2, then both algorithms yield the same value, i.e.,

$$\sum_{\ell=1}^N \angle(V_{\ell-1}, V_\ell) = \sum_{\ell=1}^N \angle(U_{\ell-1}, U_\ell).$$

Indeed, by induction we obtain $U_n^\top V_n = 0$ from the invertibility of R_n, S_n and

$$S_n^\top U_n^\top V_n R_n = U_{n-1}^\top DF_{n-1}(x_{n-1})^{-1} DF_{n-1}(x_{n-1}) V_{n-1} = U_{n-1}^\top V_{n-1} = 0.$$

The equality $\angle(V_{\ell-1}, V_\ell) = \angle(U_{\ell-1}, U_\ell)$, $\ell = 1, \dots, N$, then follows from (2.4).

For the second angular value of the 3D-Hénon map we have $s = 2 > \frac{3}{2}$. Hence, it is more efficient to iterate normal vectors and normalize them at each step rather than iterate bases of two-dimensional subspaces and perform a QR decomposition at each step. This reduces the computation time by a factor of 7. If, in addition, an explicit formula for the inverse Jacobian is used, the computation time is reduced by another factor of 3.

2.3. Application to inverted maps. Until now, we have computed angular maps only through forward iteration. Forward subspace iteration captures angular values associated with the fastest directions, while backward iteration captures those associated with the slowest directions. For this purpose we consider an invertible map $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and the autonomous system

$$(2.6) \quad y_{n+1} = f^{-1}(y_n), \quad n \in \mathbb{N}_0, \quad y_0 \in \mathbb{R}^d.$$

The associated variational equation (1.2) takes the form

$$(2.7) \quad u_{n+1} = (Df(y_{n+1}))^{-1}u_n, \quad n \in \mathbb{N}_0,$$

since $Df^{-1}(y_n) = (Df(f^{-1}(y_n)))^{-1} = (Df(y_{n+1}))^{-1}$.

Numerically, we avoid iterating the inverse map in (2.6) and instead use the orbit segment $(x_n)_{n \in \{0, \dots, N\}}$ which has already been computed forward in time. For the initial value $y_0 = x_N$, we get the solution of (2.6) by $y_n = x_{N-n}$ for $n \in \{0, \dots, N\}$. Further setting $v_k = u_{N-k}$, $k \in \{0, \dots, N\}$, the finite time variational equation (2.7) takes the form

$$v_{k-1} = (Df(x_{k-1}))^{-1}v_k, \quad k = N, \dots, 1.$$

Figure 2.4 shows the histograms of the forward and backward angular maps for the two-dimensional Hénon map f_2 in a single diagram. We choose $N = 1000$ with the remaining parameters as described above.

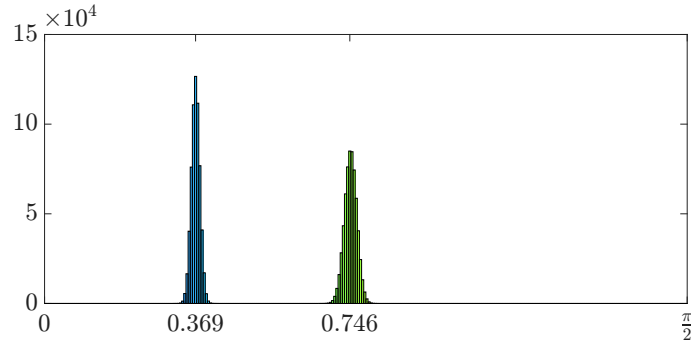


Figure 2.4: Histogram of angular maps for f_2 with $N = 1000$. The left peak corresponds to the forward computation, and the right peak to the backward computation.

We repeat this computation in Figure 2.5 for the three-dimensional Hénon map.

In the next section, we explain why forward and backward iteration can capture the entire angular spectrum for the two-dimensional Hénon map. Our numerical results suggest that, for typical trajectories approaching the attractor, this spectrum consists of the two values approximated by the peaks in Figure 2.4, with no apparent dependence on

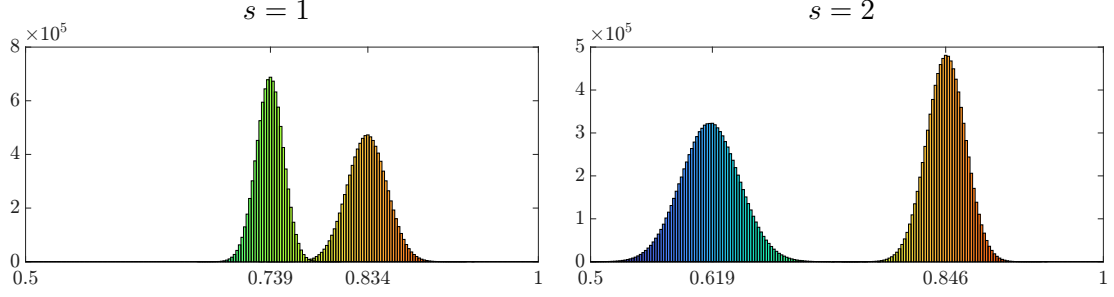


Figure 2.5: Histogram of angular maps for f_3 with $N = 1000$. For $s = 1$ the left peak corresponds to the backward computation, and the right peak to the forward computation. For $s = 2$ the left peak corresponds to the forward computation, and the right peak to the backward computation.

the initial point. Exceptional trajectories, such as fixed points and periodic orbits on the attractor, may have different angular spectra. For the three-dimensional Hénon map, however, forward and backward iteration alone miss one of the three spectral values.

3. Angular maps in discrete time and a general algorithm. In this section, we show that the algorithms discussed so far capture only a specific part of the angular spectrum and thus of the angular map. This follows from a key result of the reduction theory developed in [3], which is based on the exponential dichotomy spectrum of the variational equation (1.2). As a consequence, we will use the methods from [3] to develop an algorithm capable of computing the entire angular spectrum. However, the computational cost of this algorithm is orders of magnitude higher than the simple algorithms considered so far.

3.1. Angular range and dichotomy spectrum. The notion of an “angular range” is justified by the following Lemma taken from [5, Lemma 2.6].

Lemma 3.1. *The angular range $I_s^{\text{disc}}(V; x_0)$ is the set of accumulation points of the sequence $(\alpha_s^{\text{disc}}(n, V; x_0))_{n \in \mathbb{N}}$.*

We use the reduction theory from [3, Section 3] to analyze which part of the angular spectrum is obtained by applying the QR -like Algorithm 2 to a generic subspace $V \in \mathcal{G}(s, d)$.

Let us assume that the matrices A_n in (1.2) are uniformly bounded and have uniformly bounded inverses. According to the spectral theorem [1, Theorem 5.4] for exponential dichotomies (see also [11]), there exists an index $\varkappa \leq d$ and numbers

$$0 = \sigma_{\varkappa+1}^+ < \sigma_{\varkappa}^- \leq \sigma_{\varkappa}^+ < \sigma_{\varkappa-1}^- \leq \sigma_{\varkappa-1}^+ < \dots < \sigma_1^- \leq \sigma_1^+$$

and projectors P_n^i , $n \in \mathbb{N}$, $i = 1, \dots, \varkappa + 1$, with the following properties:

(i) For all $0 \leq m \leq n$, $i = 1, \dots, \varkappa$ there holds

$$\begin{aligned} \Phi(n, m)P_m^i &= P_n^i\Phi(n, m), & (\text{invariance}) \\ (3.1) \quad \{0\} &= \mathcal{R}(P_n^{\varkappa+1}) \subseteq \mathcal{R}(P_n^{\varkappa}) \subseteq \dots \subseteq \mathcal{R}(P_n^1) = \mathbb{R}^d, & (\text{flag of ranges}) \\ \mathbb{R}^d &= \mathcal{N}(P_n^{\varkappa+1}) \supseteq \mathcal{N}(P_n^{\varkappa}) \supseteq \dots \supseteq \mathcal{N}(P_n^1) = \{0\}. & (\text{flag of null spaces}) \end{aligned}$$

- (ii) For each $i = 1, \dots, \varkappa$ and for every $0 < \varepsilon < \sigma_i^- - \sigma_{i+1}^+$ there exists a constant $K_i(\varepsilon)$ such that for all $0 \leq m \leq n$ there holds

$$(3.2) \quad \begin{aligned} \|\Phi(n, m)P_m^{i+1}\| &\leq K_i(\varepsilon)(\sigma_{i+1}^+ + \varepsilon)^{n-m}, \quad (\text{forward dichotomy estimate}) \\ \|\Phi(m, n)(I_d - P_n^{i+1})\| &\leq K_i(\varepsilon)(\sigma_i^- - \varepsilon)^{m-n}. \quad (\text{backward dichotomy estimate}) \end{aligned}$$

The set

$$\Sigma_{\text{ED}} = \bigcup_{i=1}^{\varkappa} [\sigma_i^-, \sigma_i^+]$$

is called the dichotomy spectrum of (1.2).

Remark 3.2. In [3, Section 3] we derived the dichotomy estimates from the exponential dichotomy of a scaled operator. However, our conclusion that the estimates (3.2) hold with $\varepsilon = 0$ is only true under additional assumptions. In general, the estimates are satisfied for sufficiently small $\varepsilon > 0$ with rates and constants depending on ε . With this modification, [3, Theorem 3.4] applies to $\varepsilon = \frac{1}{4} \min_{i=1, \dots, \varkappa} (\sigma_i^- - \sigma_{i+1}^+)$ and then yields the rate $r < 1$ used in (3.5) below.

Due to the nested subspaces (3.1), the whole space can be decomposed into invariant fibers

$$(3.3) \quad \mathbb{R}^d = \bigoplus_{i=1}^{\varkappa} \mathcal{W}_n^i, \quad \mathcal{W}_n^i = \mathcal{R}(P_n^i) \cap \mathcal{N}(P_n^{i+1}), \quad i = 1, \dots, \varkappa.$$

Note that we have $\mathcal{W}_n^i = \mathcal{R}(\mathcal{P}_n^i)$ for the fiber projectors $\mathcal{P}_n^i = (I_d - P_n^{i+1})P_n^i = P_n^i - P_n^{i+1} = P_n^i(I - P_n^{i+1})$. Because of the exponential estimates (3.2), we call $\mathcal{W}_n^{\varkappa}$ the slowest fiber and $\mathcal{W}_n^1$ the fastest fiber. Further, according to [3, (3.16)] we associate to every $V \in \mathcal{G}(s, d)$ its trace spaces

$$(3.4) \quad \mathcal{T}_n(V) = \bigoplus_{i=1}^{\varkappa} \mathcal{W}_n^i(V), \quad \text{with} \quad \mathcal{W}_n^i(V) = (I_d - P_n^{i+1})(\mathcal{R}(P_n^i) \cap V),$$

which satisfy $\mathcal{W}_n^i(V) \subseteq \mathcal{W}_n^i$ as well as $\sum_{i=1}^{\varkappa} \dim(\mathcal{W}_n^i(V)) = s$.

The main result [3, Theorem 3.4] states that the angle between the iterates of V and its trace spaces $\mathcal{T}_n(\Phi(n, 0)V) = \Phi(n, 0)\mathcal{T}_0(V)$ converges to zero at an exponential rate. More precisely, for every $V \in \mathcal{G}(s, d)$ there exists a constant $C(V)$ such that for all $n \geq 0$

$$(3.5) \quad \angle(\Phi(n, 0)V, \Phi(n, 0)\mathcal{T}_0(V)) \leq C(V)r^n, \quad r = \frac{1}{2} \left(1 + \max_{i=1, \dots, \varkappa} \frac{\sigma_{i+1}^+}{\sigma_i^-} \right) < 1.$$

As noted above, the rate r is determined by the smallest gap between the spectral intervals. As a consequence of (3.5), we find that the angular range of V coincides with the angular range of its trace space at $n = 0$ [5, Theorem 2.17]:

$$(3.6) \quad I_s^{\text{disc}}(V) = I_s^{\text{disc}}(\mathcal{T}_0(V)), \quad V \in \mathcal{G}(s, d).$$

Hence, one can reduce the computation of the angular spectrum to the set of trace spaces:

$$(3.7) \quad \begin{aligned} \Sigma_s^{\text{disc}} &= \text{cl} \left[\bigcup_{V \in \mathcal{D}_0(s, d)} I_s^{\text{disc}}(V) \right], \\ \mathcal{D}_0(s, d) &= \left\{ \bigoplus_{i=1}^{\varkappa} W^i : W^i \text{ subspace of } \mathcal{W}_0^i \ (i = 1, \dots, \varkappa), \ \sum_{i=1}^{\varkappa} \dim W^i = s \right\}. \end{aligned}$$

In this way, the computational effort can be reduced significantly. In particular, if all fibers are one-dimensional, the whole angular spectrum can be computed from the angular range of at most $\binom{d}{s}$ trace spaces. This case frequently occurs with our applications below.

Further, if we choose a random initial subspace as in Algorithm 2, then we expect the algorithm to converge to the angular range of a trace space composed of the fastest fibers in (3.3). To see this, determine the unique index $j \in \{1, \dots, \varkappa\}$ with

$$(3.8) \quad \sum_{i=1}^{j-1} d_i < s \leq \sum_{i=1}^j d_i = d - \sum_{i=j+1}^{\varkappa} d_i, \quad d_i = \dim(\mathcal{W}_0^i), \quad i = 1, \dots, \varkappa.$$

From the inequality $s + \sum_{i=j+1}^{\varkappa} d_i \leq d$ we conclude that the two subspaces V and $\bigoplus_{i=j+1}^{\varkappa} \mathcal{W}_0^i$ almost surely have a trivial intersection. Then $W_0^i(V) = \{0\}$ holds for $i = j+1, \dots, \varkappa$ in (3.4) and

$$(3.9) \quad \mathcal{T}_0(V) = \bigoplus_{i=1}^j W_0^i(V) \subseteq \bigoplus_{i=1}^j \mathcal{W}_0^i.$$

Therefore, the algorithm will generate the angular range of a subspace that belongs to a sum of the fastest fibers. In case of equality on the righthand side of (3.8), we can say even more:

Proposition 3.3. *Let $0 < s < d$ and $j \in \{1, \dots, \varkappa\}$ be given. Then there exist constants $C > 0$ and $0 \leq r < 1$ such that for all $V \in \mathcal{G}(s, d)$ with*

$$(3.10) \quad s = \sum_{i=1}^j \dim(\mathcal{W}_0^i), \quad V \cap \bigoplus_{i=j+1}^{\varkappa} \mathcal{W}_0^i = \{0\},$$

the following estimates hold for the iterates of V and the angular sums from (1.4)

$$(3.11) \quad \begin{aligned} \angle(\Phi(n, 0)V, \bigoplus_{i=1}^j \mathcal{W}_n^i) &\leq \frac{C}{\theta_{\min}} r^n, \quad \forall n \geq 1, \\ |\alpha_s^{\text{disc}}(n, V) - \alpha_s^{\text{disc}}(n, \bigoplus_{i=1}^j \mathcal{W}_0^i)| &\leq \frac{C}{n\theta_{\min}}, \quad \forall n \geq 1. \end{aligned}$$

Here $\theta_{\min} = \theta_{\min}(V, \bigoplus_{i=j+1}^{\varkappa} \mathcal{W}_0^i)$ denotes the minimal principal angle between the subspaces V and $\bigoplus_{i=j+1}^{\varkappa} \mathcal{W}_0^i$. Further, the equality $I_s^{\text{disc}}(V) = I_s^{\text{disc}}(\bigoplus_{i=1}^j \mathcal{W}_0^i)$ holds.

Remark 3.4. *Note that we have $\theta_{\min} > 0$ by formula (2.3) and assumption (3.10). The estimate (3.11) refines the result of [3, Theorem 3.4] by specifying the dependence of the constant $C(V)$ on V in the asymptotic estimate (3.5). The closer V lies to the sum of the $\varkappa - j$ slow fibers, the longer it takes for the asymptotic regime to dominate.*

Proof. From the equality of dimensions in (3.10) we find equality in (3.9) as well, i.e.

$$(3.12) \quad \mathcal{T}_0(V) = \bigoplus_{i=1}^j \mathcal{W}_0^i = \mathcal{N}(P_0^{j+1}), \quad \bigoplus_{i=j+1}^{\varkappa} \mathcal{W}_0^i = \mathcal{R}(P_0^{j+1}).$$

Moreover, assumption (3.10) assures that the map

$$(I_d - P_0^{j+1})|_V : V \rightarrow \mathcal{N}(P_0^{j+1})$$

is injective, hence invertible by the equality of dimensions. Setting

$$L_0 = P_0^{j+1} \left((I_d - P_0^{j+1})|_V \right)^{-1} : \mathcal{N}(P_0^{j+1}) \rightarrow \mathcal{R}(P_0^{j+1}),$$

we can write V as a graph over $\mathcal{N}(P_0^{j+1})$, i.e.

$$(3.13) \quad v = (I_d - P_0^{j+1})v + L_0(I_d - P_0^{j+1})v, \quad \forall v \in V.$$

The characterization (2.3) of $\theta_{\min}$ and (3.13) then show for all $v \in V$

$$\sin(\theta_{\min})\|v\| \leq \text{dist}(v, \mathcal{R}(P_0^{j+1})) \leq \|(I_d - P_0^{j+1})v\|,$$

and

$$(3.14) \quad \|L_0(I_d - P_0^{j+1})v\| \leq \|v\| + \|(I_d - P_0^{j+1})v\| \leq (1 + \frac{1}{\sin(\theta_{\min})})\|(I_d - P_0^{j+1})v\|.$$

We iterate the decomposition (3.13)

$$(3.15) \quad \begin{aligned} \Phi(n, 0)v &= \Phi(n, 0)(I_d - P_0^{j+1})v \\ &\quad + \Phi(n, 0)P_0^{j+1}L_0(I_d - P_0^{j+1})\Phi(0, n)(I_d - P_n^{j+1})\Phi(n, 0)(I_d - P_0^{j+1})v. \end{aligned}$$

Let $w = \Phi(n, 0)(I_d - P_0^{j+1})v \in \mathcal{N}(P_n^{j+1})$. Using (3.14) and the dichotomy estimates (3.2) for $\Phi(n, 0)P_0^{j+1}$ and $\Phi(0, n)(I_d - P_n^{j+1})$, we obtain

$$(3.16) \quad \|\Phi(n, 0)v - w\| \leq K_j(\varepsilon)^2 \left(\frac{\sigma_{j+1}^+ + \varepsilon}{\sigma_j^- - \varepsilon} \right)^n \left(1 + \frac{1}{\sin(\theta_{\min})} \right) \|w\|.$$

Further, by the invariance of fibers we have

$$\Phi(n, 0)\mathcal{N}(P_0^{j+1}) = \Phi(n, 0) \bigoplus_{i=1}^j \mathcal{W}_0^i = \bigoplus_{i=1}^j \mathcal{W}_n^i = \mathcal{N}(P_n^{j+1}).$$

For every $w \in \mathcal{N}(P_n^{j+1})$ there exists a unique element $v \in V$ with $w = \Phi(n, 0)(I_d - P_0^{j+1})v$. Thus the characterization (2.2) and (3.16) lead to

$$\begin{aligned} \sin(\angle(\mathcal{N}(P_n^{j+1}), \Phi(n, 0)V)) &= \sup\{\text{dist}(w, \Phi(n, 0)V) : w \in \mathcal{N}(P_n^{j+1}), \|w\| = 1\} \\ &\leq K_j(\varepsilon)^2 \left(\frac{\sigma_{j+1}^+ + \varepsilon}{\sigma_j^- - \varepsilon} \right)^n \left(1 + \frac{1}{\sin(\theta_{\min})} \right). \end{aligned}$$

Using $\frac{2}{\pi}x \leq \sin(x) \leq 1$ for $0 \leq x \leq \frac{\pi}{2}$ and setting $\varepsilon = \frac{1}{4}(\sigma_j^- - \sigma_{j+1}^+)$, we obtain the first estimate in (3.11) for the constants

$$r = \frac{1}{2} \left(1 + \frac{\sigma_{j+1}^+}{\sigma_j^-} \right), \quad C_0 = \frac{1}{2}\pi^2 K_j(\varepsilon)^2.$$

With the triangle inequality for “ $\angle$ ” we then find

$$\begin{aligned} &|\alpha^{\text{disc}}(n, V) - \alpha^{\text{disc}}(n, \bigoplus_{i=1}^j \mathcal{W}_0^i)| \\ &\leq \frac{1}{n} \sum_{k=1}^n |\angle(\Phi(k-1, 0)V, \Phi(k, 0)V) - \angle(\Phi(k-1, 0)\mathcal{T}_0(V), \Phi(k, 0)\mathcal{T}_0(V))| \\ &\leq \frac{1}{n} \sum_{k=1}^n \angle(\Phi(k-1, 0)V, \Phi(k-1, 0)\mathcal{T}_0(V)) + \angle(\Phi(k, 0)V, \Phi(k, 0)\mathcal{T}_0(V)) \\ &\leq \frac{2C_0}{n\theta_{\min}} \sum_{k=0}^n r^k \leq \frac{2C_0}{n(1-r)\theta_{\min}}. \end{aligned}$$

Finally, the equality of angular ranges follows from (3.6) and (3.12). ■

In the particular case of one-dimensional fibers we can choose $j = s$ in (3.10). Since a random choice of V almost always satisfies the second condition in (3.10), we find that Algorithm 2 almost surely yields the angular range of the subspace spanned by the s fastest fibers.

On the other hand, if $s < \sum_{i=1}^j d_i$ holds in (3.8), then the reduced set of trace spaces

$$(3.17) \quad \mathcal{D}_0^j(s, d) = \left\{ \bigoplus_{i=1}^j W^i \in \mathcal{D}_0(s, d) : \sum_{i=1}^j \dim W^i = s \right\}$$

is a continuum. Therefore, we must invoke some continuous optimization method for finding the infimum and the supremum of the angular spectrum.

In Section 2.3 we used the inverse map to find the angular spectrum associated to the sum of the s slowest fibers. We derive a suitable analog of Proposition 3.3. Since we cannot invert from infinity, we invert the solution operator at any finite $N \in \mathbb{N}$ and take care that the resulting estimates are independent of N . The inverse solution operator is

$$\tilde{\Phi}(n, m) = \Phi(N - n, N - m), \quad 0 \leq n, m \leq N.$$

Note that we do not make the dependence on N explicit. It is not difficult to see that the properties (3.1), (3.2) and the decomposition (3.3) hold for $\tilde{\Phi}$ with constants independent of N and the following data ($n = 0, \dots, N$):

$$\begin{aligned} \tilde{\sigma}_i^\pm &= (\sigma_{\varkappa-i+1}^\mp)^{-1}, \quad \tilde{\mathcal{W}}_n^i = \mathcal{W}_{N-n}^{\varkappa-i+1}, \quad i = 1, \dots, \varkappa, \\ \tilde{P}_n^i &= I_d - P_{N-n}^{\varkappa-i+2}, \quad i = 1, \dots, \varkappa + 1. \end{aligned}$$

Given a subspace $\tilde{V} \in \mathcal{G}(s, d)$ at time N , the trace spaces cannot be computed in terms of the forward iteration, but are defined as follows

$$\tilde{\mathcal{T}}_{N-n}(\tilde{V}) = \bigoplus_{i=1}^{\varkappa} P_{N-n}^{\varkappa-i+1} \left(\mathcal{N}(P_{N-n}^{\varkappa-i+2}) \cap \tilde{V} \right), \quad n = 0, \dots, N.$$

Using the dichotomy estimates for the reversed system, one can show the following result.

Proposition 3.5. *Let $0 < s < d$. There exist constants $0 < r < 1$ and $C > 0$, depending only on the dichotomy estimates, such that the following holds. For every $N \in \mathbb{N}$, $j \in \{1, \dots, \varkappa\}$, and $\tilde{V} \in \mathcal{G}(s, d)$ satisfying*

$$(3.18) \quad s = \sum_{i=j}^{\varkappa} d_i, \quad \tilde{V} \cap \bigoplus_{i=1}^{j-1} \mathcal{W}_N^i = \{0\},$$

define $\tilde{\theta}_{\min} = \theta_{\min}(\tilde{V}, \bigoplus_{i=1}^{j-1} \mathcal{W}_N^i)$. Then the estimates hold

$$(3.19) \quad \begin{aligned} \angle(\Phi(N - n, N)\tilde{V}, \bigoplus_{i=j}^{\varkappa} \mathcal{W}_{N-n}^i) &\leq \frac{C}{\tilde{\theta}_{\min}} r^n, \quad 0 \leq n \leq N, \\ |\tilde{\alpha}_s^{\text{disc}}(N - n, \tilde{V}) - \tilde{\alpha}_s^{\text{disc}}(N - n, \bigoplus_{i=j}^{\varkappa} \mathcal{W}_N^i)| &\leq \frac{C}{n\tilde{\theta}_{\min}}, \quad 1 \leq n \leq N, \end{aligned}$$

where, for $V \in \mathcal{G}(s, d)$,

$$\tilde{\alpha}_s^{\text{disc}}(N - n, V) := \frac{1}{n} \sum_{k=1}^n \angle(\Phi(N - k + 1, N)V, \Phi(N - k, N)V).$$

Our numerical computations in Section 2.3 yield the quantity $\tilde{\alpha}_s^{\text{disc}}(0, \tilde{V})$, which according to (3.19) is a good approximation of

$$\tilde{\alpha}_s^{\text{disc}}(0, \bigoplus_{i=j}^{\infty} \mathcal{W}_N^i) = \frac{1}{N} \sum_{\ell=1}^N \angle(\bigoplus_{i=j}^{\infty} \mathcal{W}_\ell^i, \bigoplus_{i=j}^{\infty} \mathcal{W}_{\ell-1}^i),$$

i.e., of the angular spectrum that belongs to the slowest fibers.

3.2. Combining forward and backward iteration. Our numerical computations indicate that, for the trajectories of the Hénon models considered here, all spectral fibers are one-dimensional. Consequently, for the two-dimensional system f_2 , we compute the entire angular spectrum by applying Algorithm 1 to f_2 and f_2^{-1} .

For the three-dimensional Hénon system f_3 , however, this approach yields only the angular value of the fastest trace space $\mathcal{W}_n^1$ (via forward iteration) and the slowest trace space $\mathcal{W}_n^3$ (via backward iteration). Furthermore, using Algorithm 3, we have already computed the normal vector $\eta_n^{1,2}$, which is orthogonal to $\mathcal{W}_n^1 \oplus \mathcal{W}_n^2$ via forward iteration as well as the normal vector $\eta_n^{2,3}$ via backward iteration, which is orthogonal to $\mathcal{W}_n^2 \oplus \mathcal{W}_n^3$. The cross product $\omega_n^2 = \eta_n^{1,2} \times \eta_n^{2,3}$ of these two vectors spans the second trace space, i.e., $\text{span}(\omega_n^2) = \mathcal{W}_n^2$ for $n \in \{0, \dots, N\}$.

In summary, for $i \neq j \in \{1, 2, 3\}$ and $n \in \{0, \dots, N\}$, the algorithms described above yield vectors ω_n^i and $\eta_n^{i,j} = \omega_n^i \times \omega_n^j$ such that

$$\text{span}(\omega_n^i) = \mathcal{W}_n^i \quad \text{and} \quad \text{span}(\eta_n^{i,j}) = (\mathcal{W}_n^i \oplus \mathcal{W}_n^j)^\perp.$$

Using these subspaces together with (2.4), we compute the approximate angular spectra $\Sigma_{s,N}^{\text{disc}}$ without requiring any further iterations:

$$\begin{aligned} \Sigma_{1,N}^{\text{disc}} &= \{\tilde{\alpha}_1^{\text{disc}}(0, \mathcal{W}_N^i) : i \in \{1, 2, 3\}\} = \left\{ \frac{1}{N} \sum_{\ell=1}^N \angle(\omega_\ell^i, \omega_{\ell-1}^i) : i \in \{1, 2, 3\} \right\}, \\ \Sigma_{2,N}^{\text{disc}} &= \{\tilde{\alpha}_2^{\text{disc}}(0, \mathcal{W}_N^i \oplus \mathcal{W}_N^j) : i, j \in \{1, 2, 3\}, i < j\} \\ &= \left\{ \frac{1}{N} \sum_{\ell=1}^N \angle(\eta_\ell^{i,j}, \eta_{\ell-1}^{i,j}) : i, j \in \{1, 2, 3\}, i < j \right\}. \end{aligned}$$

Figure 3.1 shows the resulting histograms. In [3, Section 4.2.4], the numerical results for both Hénon models were obtained using the general algorithm described in Section 3.3. Here, we reproduce these results using forward and backward iterations without having to compute the dichotomy spectrum.

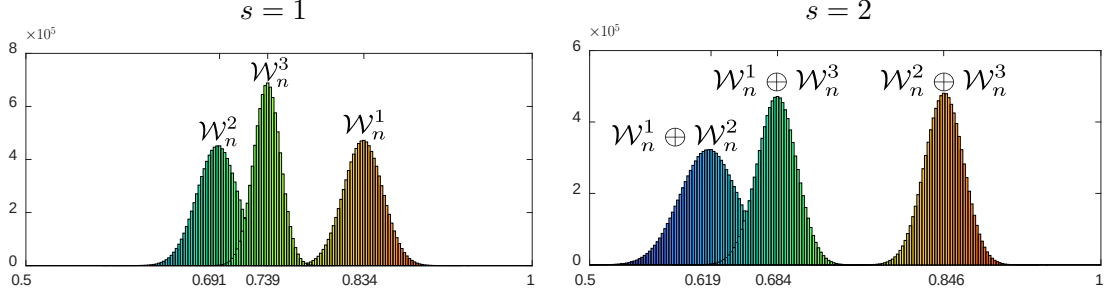


Figure 3.1: Histograms of the angular map for the three-dimensional Hénon map f_3 , computed with $N = 1000$, for $s = 1$ (left) and $s = 2$ (right). Each peak is labeled with its associated trace space.

In Figure 3.2, we vary the length N of the computed orbit over the range $10^2 \leq N \leq 10^7$ while keeping the initial point fixed at $x_0 = (0.2, 0.1, 0)^\top$. The orbit starting at x_0 approaches the attractor of the three-dimensional Hénon system. Our numerical results suggest that the six angular sums converge to angular spectral values as N increases.

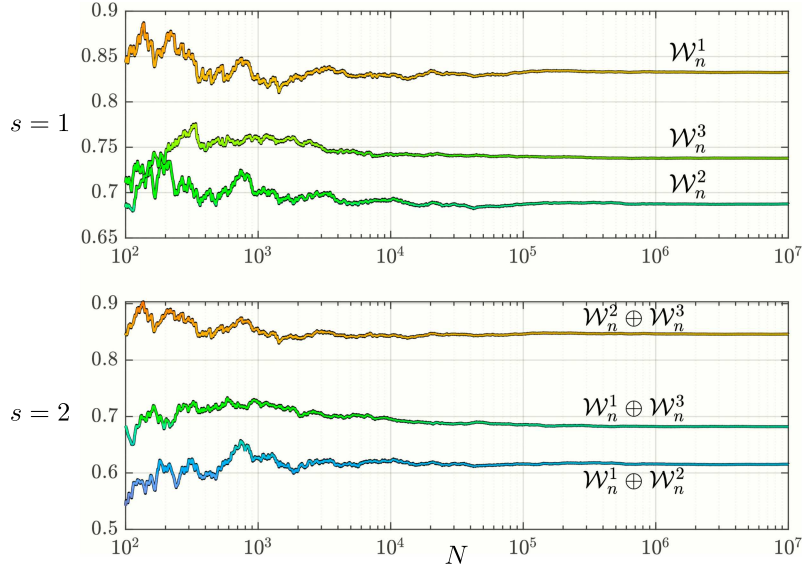


Figure 3.2: Angular values for the fixed initial point $x_0 = (0.2, 0.1, 0)^\top$ and orbit lengths ranging from $N = 10^2$ to $N = 10^7$, with $s = 1$ in the upper panel and $s = 2$ in the lower panel. The color coding is the same as in Figure 3.1.

3.3. The general algorithm in discrete time. In Section 3.2 we approximated the entire angular spectrum along the trajectories considered for the three-dimensional Hénon system using the QR-like Algorithm 2. This worked since all fibers are one-dimensional and since we employed techniques like inversion of the system and forming orthogonal complements or intersections of fibers. For systems of dimension $d > 3$, combining forward and backward iteration becomes more involved. For these systems, as well as for systems

with spectral fibers of dimension ≥ 2 , we use the trace-space approach described below to obtain a complete picture of the angular spectrum and the corresponding angular map.

According to Section 3.1 we can achieve this by computing the angular ranges (1.5) for all trace spaces $V \in \mathcal{D}_0(s, d)$; see (3.7). This in turn requires to first compute the dichotomy spectrum and its associated invariant foliation. Then in a second step one can form the possible trace spaces and compute their angular spectral values. Such an algorithm was set up in [5, Section 4.2] under the assumption that all spectral fibers are one- or two-dimensional. The algorithm proves to be quite expensive if the set of trace spaces is not finite but consists of one- or two-dimensional continua (see the remarks following (3.17)). In such cases, we limit ourselves to computing lower and upper bounds of the set $\Sigma_{s,N}^{\text{disc}}(x_0) = \{\alpha_s^{\text{disc}}(N, V; x_0) : V \in \mathcal{D}_0(s, d; x_0)\}$, given by

$$\Sigma_{s,N}^{\text{disc},\min}(x_0) = \inf \Sigma_{s,N}^{\text{disc}}(x_0), \quad \Sigma_{s,N}^{\text{disc},\max}(x_0) = \sup \Sigma_{s,N}^{\text{disc}}(x_0).$$

Here $\mathcal{D}_0(s, d; x_0)$ denotes the set of trace spaces associated with the variational equation along the trajectory starting at x_0 .

We consider maps $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and let $\mathcal{B} = \times_{j=1}^d [a_j, b_j]$ be a d -dimensional rectangular box. By L we denote the resolution of our grid and define $\mathcal{L} = \{1, \dots, L\}$ and $h_i = \frac{b_i - a_i}{L}$ for $i = 1, \dots, d$. For every multi-index

$$\mathbf{i} = (i_1, \dots, i_d) \in \mathcal{L}^d,$$

we denote the corresponding grid box by

$$\mathcal{B}_{\mathbf{i}} := \times_{j=1}^d [a_j + (i_j - 1)h_j, a_j + i_j h_j].$$

The general algorithm reads:

Algorithm 4 Angular maps for $s \in \{1, 2\}$ and general d

```

1: for  $\mathbf{i} \in \mathcal{L}^d$  do
2:   Choose the midpoint  $x_0 \in \mathcal{B}_{\mathbf{i}}$ 
3:   for  $n = 1, \dots, N$  do
4:      $x_n = F_{n-1}(x_{n-1})$ 
5:      $A_{n-1} = DF_{n-1}(x_{n-1})$ 
6:   Compute the angular spectral values  $\Sigma_{s,N}^{\text{disc},\min}$  and  $\Sigma_{s,N}^{\text{disc},\max}$  for the linear difference
   equation  $u_{n+1} = A_n u_n$ ,  $n = 0, \dots, N-1$ , using the algorithm from [5, Section 4.2]
   with buffer intervals of length  $B$ 
7:   Color the box  $\mathcal{B}_{\mathbf{i}}$  according to  $\Sigma_{s,N}^{\text{disc},\min}$  and  $\Sigma_{s,N}^{\text{disc},\max}$ , respectively

```

The role of the buffer intervals is explained below. In Algorithm 4, we color the initial box of a starting point according to the minimal resp. the maximal angular value of the corresponding trajectory. These values depend on the finite length N of the trajectory, but we expect them to converge as $N \rightarrow \infty$.

The major work of the algorithm occurs in step 6, where the angular spectral values $\Sigma_{s,N}^{\text{disc},\min}$ and $\Sigma_{s,N}^{\text{disc},\max}$ are computed. It is essential that the algorithm in [5, Section 4.2]

computes these values for the trace spaces V in $\mathcal{G}(s, d)$ rather than for all elements of $\mathcal{G}(s, d)$; see (3.7). The computation of trace spaces requires most of the computational effort. In some cases, such as the Lorenz system treated in Section 4.4, it turns out that only finitely many trace spaces (in fact, very few) are analyzed by the algorithm.

Approximation errors occurring in the computation of trace spaces decay exponentially fast from the boundary of the computational interval toward its center; see [8], [5, Section 4.2]. Thus, we introduce left and right buffer intervals and utilize only the accurate data between these buffer intervals for computing the angular maps.

3.4. The area-preserving Hénon map. We consider an area-preserving version of Hénon's map (see [9])

$$(3.20) \quad f_a : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} a - x_1^2 + x_2 \\ x_1 \end{pmatrix},$$

with parameter $a = 0.3$. As we show below, trajectories on certain invariant curves surrounding the elliptic periodic points have dichotomy spectrum $\Sigma_{\text{ED}} = \{1\}$. For these trajectories, the only spectral fiber is $\mathbb{R}^2$, so determining the extremal angular spectral values requires optimization over all one-dimensional subspaces.

For a numerical experiment we choose $\mathcal{B} = [-1, 1]^2$. Figure 3.3 shows the midpoints of an $L \times L$ grid, with $L = 10^3$, whose trajectories remain in $\mathcal{B}$ throughout $N = 2000$ iterations.

For every grid point, the left panel displays the computed minima, whereas the right panel displays the computed maxima. The corresponding histograms are shown in the lower row. Figure 3.4 displays the pointwise difference between the maximal and minimal angular spectral values shown in Figure 3.3. A logarithmic color scale is used to make small differences more visible.

Let us add a few analytical findings for this system with $0 < a < 1$, which are in accordance with the numerical observations. Area preservation follows from $|\det(Df_a(x))| = 1$ for all $x \in \mathbb{R}^2$. Then there are two hyperbolic fixed points $x_{\pm} = (\pm\sqrt{a}, \pm\sqrt{a})^T$ for which $Df_a(x_{\pm})$ has eigenvalues of modulus $\sqrt{a} + \sqrt{1+a} > 1$ and $-\sqrt{a} + \sqrt{1+a} < 1$. Further, there is a 2-periodic orbit

$$p_1 = \begin{pmatrix} \sqrt{a} \\ -\sqrt{a} \end{pmatrix}, p_2 = f_a(p_1) = -p_1$$

located at the centers of the two colored domains in Figure 3.3. The points $p_{1,2}$ are elliptic fixed points of f_a^2 , since $D(f_a^2)(p_1) = Df_a(p_2)Df_a(p_1)$ has eigenvalues

$$\lambda_+ = 1 - 2a + 2i\sqrt{a(1-a)}, \quad \lambda_- = \overline{\lambda_+}, \quad |\lambda_{\pm}| = 1.$$

Therefore, the linearization about the two-periodic orbit $p_1, p_2, p_1, p_2, \dots$ has no exponential dichotomy. One can say much more about trajectories starting near these elliptic points by applying Moser's Twist Theorem [10, Ch. II.4].

Proposition 3.6. *Let $a \in (0, 1) \setminus \{\frac{1}{2}, \frac{3}{4}, \frac{5}{8}\}$. In every neighborhood of p_1 there are invariant curves of f_a^2 , on which f_a^2 is C^1 -conjugate to a rigid rotation with some diophantine frequency ω , i.e., $|\frac{\omega}{2\pi} - \frac{p}{q}| \geq Cq^{-\ell}$ for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and some $\ell > 2$, $C > 0$. Initial*

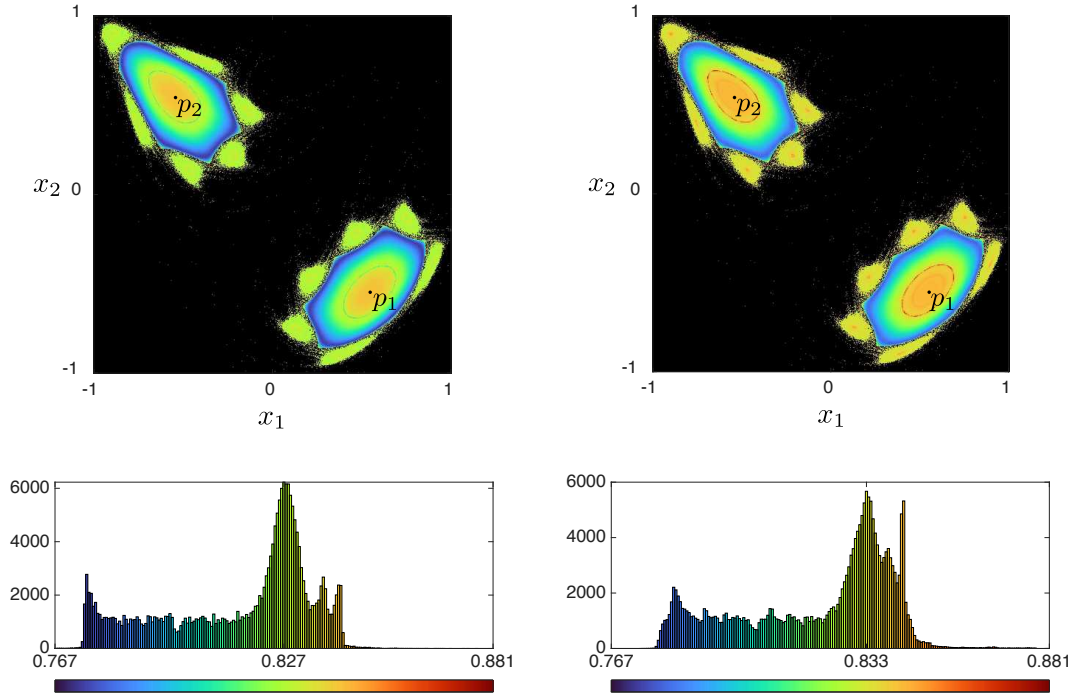


Figure 3.3: Angular maps of the minimum (left) and maximum (right) angular spectral values for the area-preserving Hénon map (3.20). The corresponding histograms are shown in the lower row.

points on these curves lead to trajectories for which the linearization of f_a has dichotomy spectrum $\Sigma_{\text{ED}} = \{1\}$.

Proof. First note that the assumption $a \in (0, 1) \setminus \{\frac{1}{2}, \frac{3}{4}\}$ excludes the $1 : k$ resonance $\lambda_+^k = 1$ for $k = 1, \dots, 4$. Then there are local coordinate changes under which f_a^2 near p_1 can be put into Birkhoff normal form [10, Ch. II.4, Theorem 2.12], given in complex notation as

$$\mathbb{C} \ni z \mapsto \lambda_+ z(1 + iB_1(a)|z|^2) + \mathcal{O}(|z|^4).$$

Note that the variable z can still be scaled by a nonzero complex number, so that the first Birkhoff coefficient $B_1(a)$ is only unique up to a positive multiple. With a suitable normalization, a symbolic toolbox yields the expression

$$B_1(a) = \frac{8a - 5}{4\sqrt{a}(1 - a)(4a - 3)}.$$

Hence, we have $B_1(a) \neq 0$ for $a \neq \frac{5}{8}$, and the Twist Theorem [10, Ch. II.4, Theorem 2.11] applies. Let Γ be one of the emerging invariant curves. Then there exists a C^1 -immersion $H : S^{2\pi} = \mathbb{R}/(2\pi\mathbb{Z}) \rightarrow \mathbb{R}^2$ such that $H(S^{2\pi}) = \Gamma$ and

$$(3.21) \quad f_a^2 \circ H = H \circ R_\omega, \quad R_\omega(\theta) = \theta + \omega \bmod 2\pi,$$

for some ω satisfying the diophantine condition.

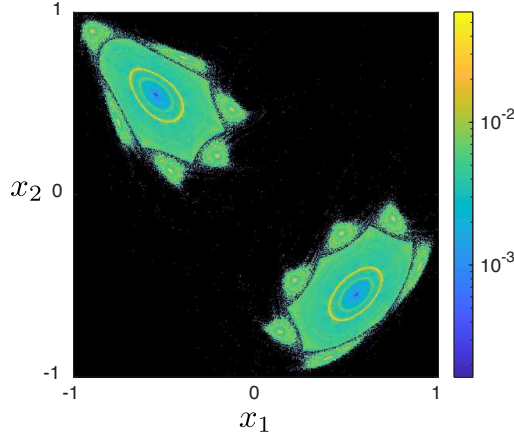


Figure 3.4: Pointwise difference between the maximal and minimal angular spectral values shown in Figure 3.3. The colors are displayed on a logarithmic scale.

For $\theta \in S^{2\pi}$ the vectors $\tau(\theta) = H'(\theta)$ and $\rho(\theta) = J\tau(\theta)\|\tau(\theta)\|^{-2}$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, form an orthogonal system at $H(\theta)$ satisfying $\det(\tau(\theta) \ \rho(\theta)) = 1$. By differentiating (3.21) one finds

$$(3.22) \quad D(f_a^2)(H(\theta)) \begin{pmatrix} \tau(\theta) & \rho(\theta) \end{pmatrix} = \begin{pmatrix} \tau(\theta + \omega) & \rho(\theta + \omega) \end{pmatrix} \begin{pmatrix} 1 & c(\theta) \\ 0 & d(\theta) \end{pmatrix}.$$

Taking the determinant, we obtain $d(\theta) \equiv 1$ and

$$c(\theta) = \frac{\tau(\theta + \omega)^\top D(f_a^2)(H(\theta))\rho(\theta)}{\|\tau(\theta + \omega)\|^2}.$$

By induction the representation of the Jacobian in suitable coordinates (3.22) leads to

$$(3.23) \quad D(f_a^{2n})(H(\theta)) \begin{pmatrix} \tau(\theta) & \rho(\theta) \end{pmatrix} = \begin{pmatrix} \tau(\theta + n\omega) & \rho(\theta + n\omega) \end{pmatrix} \begin{pmatrix} 1 & \sum_{j=0}^{n-1} c(\theta + j\omega) \\ 0 & 1 \end{pmatrix}.$$

Since $c(\theta)$ and the matrix $\begin{pmatrix} \tau(\theta) & \rho(\theta) \end{pmatrix}$ as well as its inverse are uniformly bounded, we obtain a constant $C > 0$ such that

$$\|D(f_a^{2n})(H(\theta))\| \leq C(1 + |n|) \quad \text{for all } n \in \mathbb{Z}, \theta \in S^{2\pi}.$$

Thus, the variational equation for f_a^2 has no exponentially growing solutions in forward or backward time. If f_a has such a solution, it will have the same behavior for f_a^2 , and therefore we conclude $\Sigma_{\text{ED}} = \{1\}$. ■

Remark 3.7. In (3.23), the Jacobians $D(f_a^{2n})$ are represented in local coordinates by a shear matrix $\begin{pmatrix} 1 & c_n \\ 0 & 1 \end{pmatrix}$, where $c_n = \sum_{j=0}^{n-1} c(\theta + j\omega)$. On sufficiently small Moser invariant curves, the nonzero twist implies linear growth of these shear sums. Consequently, nontangential directions approach the tangent direction at a rate $\mathcal{O}(n^{-1})$. In numerical computations, we also observe that one-dimensional subspaces approach each other at a polynomial rate. However, we do not exploit this behavior and retain the original optimization procedure.

4. Angular maps for continuous flows and a general algorithm. In this section we consider general nonautonomous systems in continuous time

$$(4.1) \quad \dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}^d,$$

defined by some vector field $f \in C^1(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$. Extending the theory from [4], we set up proper notions of angular spectrum and angular map for systems obtained by linearizing (4.1) at a solution. Further, we show that the angular spectrum of the discretized linear system converges to its continuous counterpart when the step-size goes to zero.

4.1. Definition of angular spectrum and angular maps. Let $\Psi(t_1, t_0; x_0)$ denote the solution operator of (4.1), which maps the initial value $x(t_0) = x_0$ to the solution of (4.1) at time t_1 . We choose $t_0 = 0$ and note that $\Psi(t, 0; x_0)$ exists on an open maximal interval $J = J(x_0) \subseteq \mathbb{R}$ containing 0. The associated variational equation is

$$(4.2) \quad \dot{u}(t) = A(t; x_0)u(t), \quad t \in J, \quad A(t; x_0) = \frac{\partial f}{\partial x}(t, \Psi(t, 0; x_0)).$$

By $u(t, s; x_0, u_0)$ we denote the solution of this linear system with initial condition $u(s) = u_0$ for some $s \in J$ and $u_0 \in \mathbb{R}^d$. Since the solution depends linearly on u_0 , this defines the solution operator $\Phi(\cdot, \cdot; x_0) : J(x_0) \times J(x_0) \rightarrow \mathbb{R}^{d,d}$ via

$$\Phi(t, s; x_0)u_0 = u(t, s; x_0, u_0).$$

In [4, Def. 4.3] we defined angular values for linear time-varying systems of the type (4.2) using the following quantities for $T > 0$ and $V \in \mathcal{G}(s, d)$:

$$(4.3) \quad \alpha_s^{\text{cont}}(T, V; x_0) = \frac{1}{T} \int_0^T \|(I_d - P_{\Phi(t, 0; x_0)V})A(t; x_0)P_{\Phi(t, 0; x_0)V}\| dt,$$

where $\|\cdot\|$ is the spectral norm and $P_{\Phi(t, 0; x_0)V}$ denotes the orthogonal projector onto the subspace $\Phi(t, 0; x_0)V \in \mathcal{G}(s, d)$.

In what follows we consider a domain $\mathcal{B} \subseteq \mathbb{R}^d$ such that the solutions of (4.1) exist for all $t \geq 0$ and for all $x_0 \in \mathcal{B}$. Moreover, we assume that we have a uniform bound

$$\|A(t; x_0)\| \leq C, \quad \forall t \geq 0, x_0 \in \mathcal{B}.$$

This implies the bound $\alpha_s^{\text{cont}}(T, V; x_0) \leq C$ for the values in (4.3) and hence that the limit superior and limit inferior are finite. This justifies the following analog of Definition 1.1.

Definition 4.1. For any $V \in \mathcal{G}(s, d)$ and $x_0 \in \mathcal{B}$ the interval

$$(4.4) \quad I_s^{\text{cont}}(V; x_0) = \left[\liminf_{T \rightarrow \infty} \alpha_s^{\text{cont}}(T, V; x_0), \overline{\lim}_{T \rightarrow \infty} \alpha_s^{\text{cont}}(T, V; x_0) \right]$$

is called the s -dimensional **angular range** of V for the linear system (4.2). The **angular spectrum** of dimension $s \in \{1, \dots, d\}$ is defined by

$$(4.5) \quad \Sigma_s^{\text{cont}}(x_0) = \text{cl} \left[\bigcup_{V \in \mathcal{G}(s, d)} I_s^{\text{cont}}(V; x_0) \right].$$

The map

$$\Sigma_s^{\text{cont}} : \begin{array}{ccc} \mathcal{B} & \rightarrow & \mathcal{A}(\mathbb{R}) \\ x_0 & \mapsto & \Sigma_s^{\text{cont}}(x_0) \end{array}$$

is called the **angular map** of the system (4.2) in the domain $\mathcal{B} \subseteq \mathbb{R}^d$. Here $\mathcal{A}(\mathbb{R})$ denotes the collection of nonempty compact subsets of $\mathbb{R}$.

The outer angular values defined in [4, Def. 4.3] depend on x_0 in our setting. They are related to the angular spectrum as follows:

$$\vartheta_s^{\text{inf}, \lim}(x_0) = \inf \Sigma_s^{\text{cont}}(x_0), \quad \vartheta_s^{\text{sup}, \overline{\lim}}(x_0) = \sup \Sigma_s^{\text{cont}}(x_0).$$

The analog of Lemma 3.1 in the continuous case is the following.

Lemma 4.2. *The value $\theta \in \mathbb{R}$ belongs to $I_s^{\text{cont}}(V; x_0)$ if and only if there exists a sequence $T_n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} \alpha_s^{\text{cont}}(T_n, V; x_0) = \theta$.*

Proof. If θ is one of the endpoints of $I_s^{\text{cont}}(V; x_0)$, then the assertion is clear. Otherwise, we find for every $n \in \mathbb{N}$ some $T_-(n) \geq n$ with $\alpha_s^{\text{cont}}(T_-(n), V; x_0) < \theta$. And there exist numbers $T_+(n) \geq T_-(n)$ with $\alpha_s^{\text{cont}}(T_+(n), V; x_0) > \theta$. The function $\alpha_s^{\text{cont}}(T, V; x_0)$ is continuous w.r.t. T ; hence, there exist $T_n \in [T_-(n), T_+(n)]$ with $\alpha_s^{\text{cont}}(T_n, V; x_0) = \theta$ by the intermediate value theorem. Since $T_n \rightarrow \infty$, this proves our assertion. ■

The integral expression in (4.3) was obtained in [4, Section 4] as the limit

$$\alpha_s^{\text{cont}}(T, V; x_0) = \lim_{h \rightarrow 0} \alpha_s^{\text{step}}(h, T, V; x_0)$$

of angular averages α_s^{step} which are formed with a stepsize $h = \frac{T}{N}$ as follows:

$$(4.6) \quad \alpha_s^{\text{step}}(h, T, V; x_0) = \frac{1}{T} \sum_{j=1}^N \angle(\Phi((j-1)h, 0; x_0)V, \Phi(jh, 0; x_0)V), \quad h = \frac{T}{N}.$$

Note that the term $\frac{1}{T} = \frac{1}{hN}$ in (4.6) carries an additional factor $\frac{1}{h}$ when compared to (1.4). The reason is that the quantity to be averaged over time is the rate of change of the principal angles per unit time, rather than the principal angles themselves. In (4.6) the time unit is h , whereas the discrete formula (1.4) tacitly assumes that one step corresponds to a time unit of length 1. In Section 4.5 below we provide a precise estimate of the error $|\alpha_s^{\text{cont}}(T, V; x_0) - \alpha_s^{\text{step}}(h, T, V; x_0)|$. This will again justify that the sum in (4.6) is scaled correctly.

4.2. The general algorithm for flows. Let $T > 0$ be the length of the time interval and let $h = \frac{T}{N}$ be the step size. We use the multi-index notation from Section 3.3.

In Algorithm 5 we use $\text{RK}(f, h, x_n)$ to denote one step of the fourth-order explicit Runge-Kutta scheme with step size h , applied to (4.1) with initial value x_n . For nonautonomous ODEs, the starting time nh is understood implicitly in the notation $\text{RK}(f, h, x_n)$. Correspondingly, we consider the discrete system

$$x_n = \text{RK}(f, h, x_{n-1}), \quad n = 1, \dots, N$$

Algorithm 5 Angular maps for $s \in \{1, 2\}$ and general d

```

1: for  $i \in \mathcal{L}^d$  do
2:   Choose the midpoint  $x_0 \in \mathcal{B}_i$ 
3:   for  $n = 1, \dots, N$  do
4:      $x_n = RK(f, h, x_{n-1})$ 
5:      $A_{n-1} = \frac{\partial RK}{\partial x}(f, h, x_{n-1})$ 
6:   Compute the angular spectral values  $\Sigma_{s,N}^{\text{disc},\min}$  and  $\Sigma_{s,N}^{\text{disc},\max}$  for the linear difference
   equation  $u_{n+1} = A_n u_n$ ,  $n = 0, \dots, N-1$ , using the algorithm from [5, Section 4.2]
   with buffer intervals of length  $B$ 
7:   Normalize  $\Sigma_s^{\text{RK},\min} := h^{-1} \Sigma_{s,N}^{\text{disc},\min}$  and  $\Sigma_s^{\text{RK},\max} := h^{-1} \Sigma_{s,N}^{\text{disc},\max}$ 
8:   Color the box  $\mathcal{B}_i$  according to  $\Sigma_s^{\text{RK},\min}$  and  $\Sigma_s^{\text{RK},\max}$ , respectively

```

and the solution operator $\Phi_h(n, m; x_0)$, $n \geq m \geq 0$, of the linearized system

$$u_{n+1} = \frac{\partial RK}{\partial x}(f, h, x_n) u_n, \quad n = 0, \dots, N-1.$$

Then we replace the flow Φ in (4.6) by the solution operator Φ_h , i.e., we compute

$$\alpha_s^{\text{RK}}(h, T, V; x_0) = \frac{1}{T} \sum_{j=1}^N \angle(\Phi_h(j-1, 0; x_0)V, \Phi_h(j, 0; x_0)V).$$

As in Definition 1.1 we define the s -dimensional **approximate angular spectrum** by

$$\Sigma_s^{\text{RK}}(h, T; x_0) = \text{cl} \{ \alpha_s^{\text{RK}}(h, T, V; x_0) : V \in \mathcal{G}(s, d) \}.$$

Its extremal values are given by

$$\Sigma_s^{\text{RK},\min}(h, T; x_0) = \inf \Sigma_s^{\text{RK}}(h, T; x_0), \quad \Sigma_s^{\text{RK},\max}(h, T; x_0) = \sup \Sigma_s^{\text{RK}}(h, T; x_0),$$

and

$$\Sigma_s^{\text{RK},\text{diff}}(h, T; x_0) = \Sigma_s^{\text{RK},\max}(h, T; x_0) - \Sigma_s^{\text{RK},\min}(h, T; x_0).$$

In practical computations the whole set $\mathcal{G}(s, d)$ is replaced by approximate trace spaces which are computed from an approximate dichotomy spectrum obtained at finite time. In the following we do not discuss the errors induced by these additional approximations.

4.3. Application to an autonomous planar flow. Our first example in continuous time is inspired by a nonautonomous fluid flow model introduced in [6]. We consider the two-dimensional autonomous ODE model, defined on $\mathcal{B} = [0, 2\pi] \times [0, \pi]$

$$(4.7) \quad \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos(u_2^2) - \sin(u_1) \cos(u_2) + \frac{\sin(u_1)}{\left((\sin(u_1) \sin(u_2) - \frac{u_2}{2} + \frac{\pi}{4})^2 + 1 \right)^4} \\ \cos(u_1) \sin(u_2) \end{pmatrix}.$$

The left and right boundaries of $\mathcal{B}$ are identified, so that the system is considered on $(\mathbb{R}/2\pi\mathbb{Z}) \times [0, \pi]$.

Note that $\sin(0) = \sin(\pi) = 0$; thus solutions cannot leave $\mathcal{B}$ through the upper and lower boundaries.

We choose $N = 2000$ and $h = \frac{1}{20}$ and apply five successive steps of the classical fourth-order Runge–Kutta scheme with step size $\frac{1}{100}$ to obtain a numerical approximation of the h -flow. In Algorithm 5, RK denotes this composition, and $\frac{\partial \text{RK}}{\partial x}$ its derivative. We then apply this algorithm on an $L \times L$ grid, with $L = 10^3$. Numerical tests indicate that for almost all initial points, the only spectral fiber is the entire space $\mathbb{R}^2$. Thus, we omit the computationally expensive calculation of the dichotomy spectrum and its corresponding trace spaces and solve only optimization problems in line 6 of Algorithm 5. Figure 4.1 shows the resulting angular maps in the upper row and corresponding histograms, using the same color scheme, in the lower row. Figure 4.2 displays the pointwise difference between the maximal and minimal angular spectral values shown in Figure 4.1. A logarithmic color scale is used to make small differences more visible.

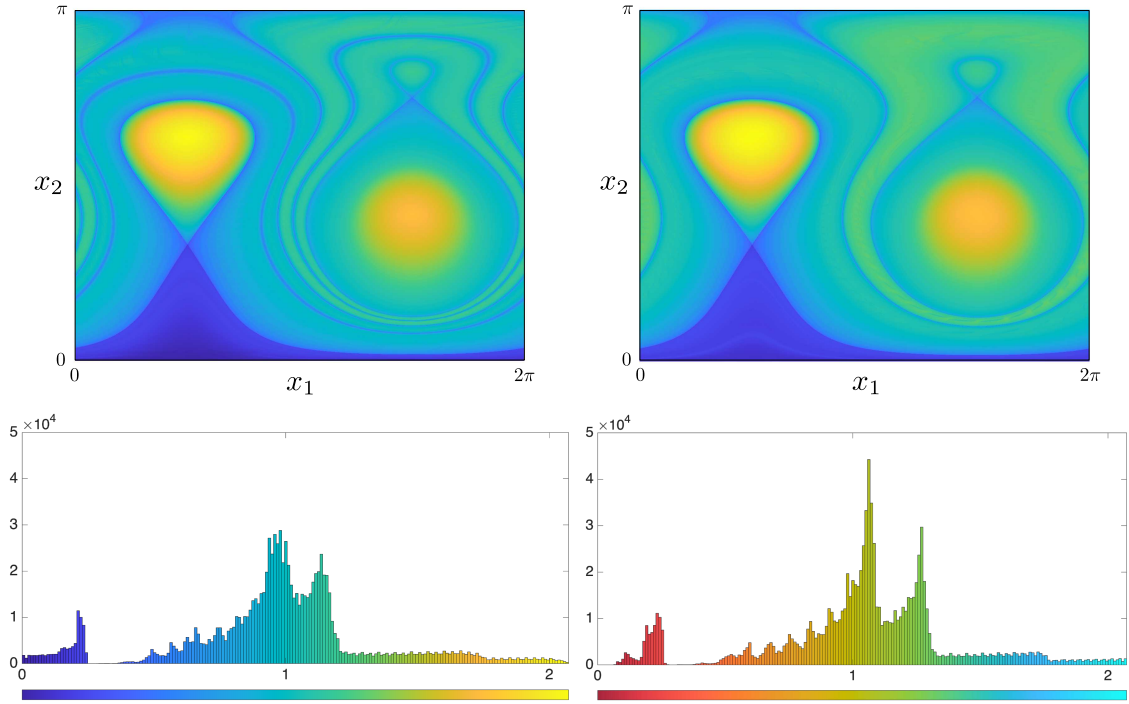


Figure 4.1: Angular maps of the minimum (left) and maximum (right) angular spectral values for (4.7). The corresponding histograms are shown in the lower row.

4.4. Application to the 3D-Lorenz system. Our second example is the well-known three-dimensional Lorenz system

$$(4.8) \quad \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \sigma(u_2 - u_1) \\ \rho u_1 - u_2 - u_1 u_3 \\ u_1 u_2 - \beta u_3 \end{pmatrix} \quad \text{with parameters } \sigma = 10, \rho = 28, \beta = \frac{8}{3}.$$

We first take the h -flow of the Lorenz system with $h = \frac{1}{20}$ and compute an orbit that is close to the Lorenz attractor. For the initial conditions tested, the computed finite-time

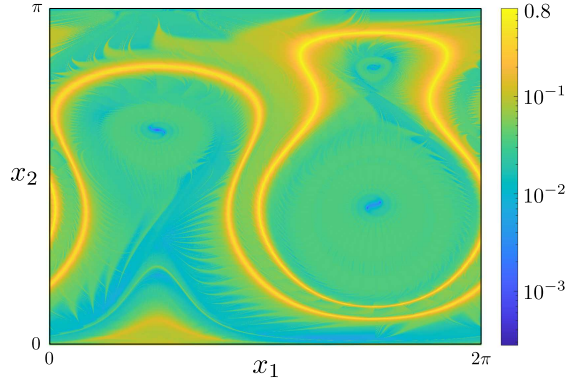


Figure 4.2: Pointwise difference between the maximal and minimal angular spectral values shown in Figure 4.1. The colors are displayed on a logarithmic scale.

spectral intervals of the dichotomy spectrum show little dependence on the initial point. Our numerical procedure yields three separated spectral intervals near 0.48, 1, and 1.05.

Next, we take a grid of size L^3 with $L = 100$ in $\mathcal{B} = [-20, 30] \times [-20, 30] \times [-10, 50]$. For all initial conditions chosen at the midpoints of the grid cells, the computed orbits eventually approach the Lorenz attractor. Thus, we observe only one-dimensional spectral fibers in the Lorenz system. This allows us to avoid the costly computation of the dichotomy spectrum and its associated trace spaces at every grid point. Instead, as in Section 3.2, we approximate the relevant trace spaces by combining forward and backward iterations. Figure 4.3 shows the three-dimensional angular maps for $N = 200$ and $N = 2000$, together with the corresponding histograms.

For $N = 2000$, we compute all angular spectral values for $s = 1$ and $s = 2$. The corresponding histograms are shown in Figure 4.4. We present only the histograms and label each peak with the trace space in which the corresponding angular spectral value is attained.

4.5. Convergence of angular spectra for time-discretized systems. In the following we analyze the approximation error

$$e(h, T, V) = |\alpha_s^{\text{step}}(h, T, V) - \alpha_s^{\text{cont}}(T, V)|$$

of the terms in (4.6) and (4.3). To simplify notation, we omit the dependence on the initial data x_0 , i.e., we write

$$\alpha_s^{\text{step}}(h, T, V) = \alpha_s^{\text{step}}(h, T, V; x_0), \quad \alpha_s^{\text{cont}}(T, V) = \alpha_s^{\text{cont}}(T, V; x_0).$$

In Remark 4.7 we will indicate when all the estimates hold uniformly in $x_0 \in \mathcal{B}$. Our goal is to prove the following theorem.

Theorem 4.3. *Let $A \in C_b^1([0, \infty), \mathbb{R}^{d,d})$, i.e., $A(\cdot)$ and $\dot{A}(\cdot)$ are continuous and bounded. Then there exist constants $C, h_0 > 0$ such that the estimate*

$$e(h, T, V) \leq Ch$$

holds for all $V \in \mathcal{G}(s, d)$, for all $T > 0$ and for all $h = \frac{T}{N} \leq h_0$, where $N \in \mathbb{N}$.

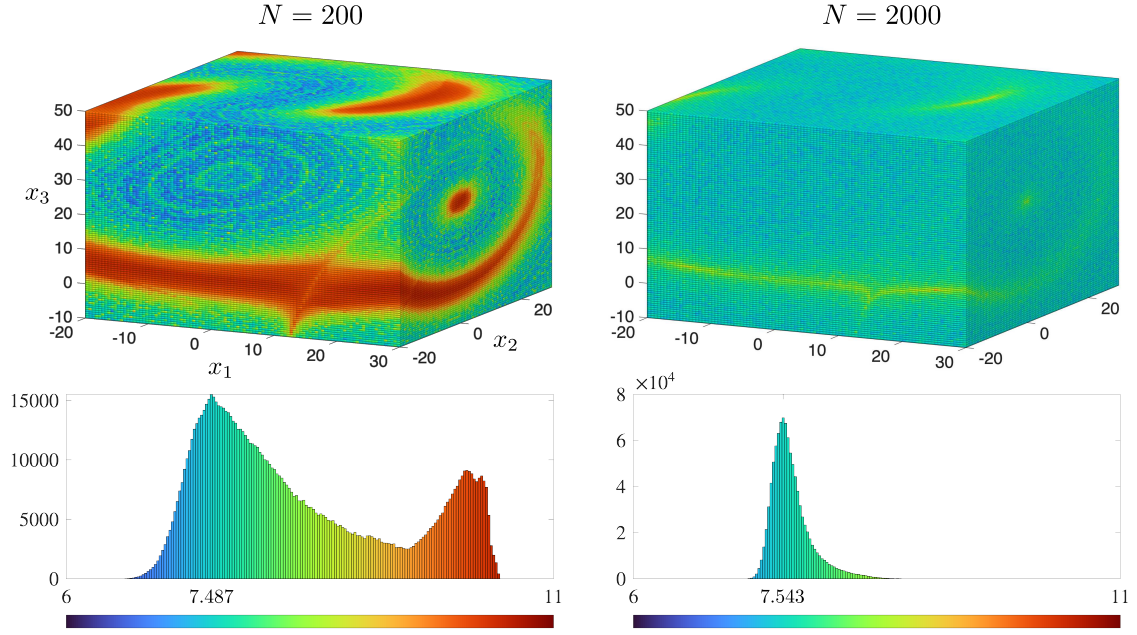


Figure 4.3: Spectral angular values for $s = 1$ computed by forward iteration of the h -flow of the Lorenz system (4.8), with $h = \frac{1}{20}$. The results for $N = 200$ and $N = 2000$ are shown on the left and right, respectively, together with the corresponding histograms.

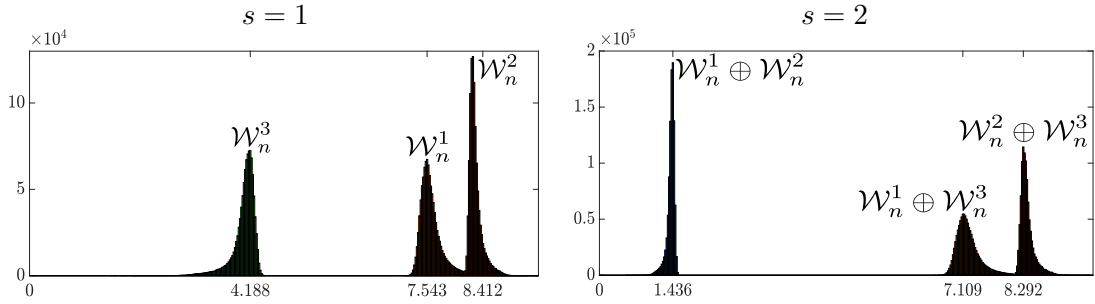


Figure 4.4: Histograms of the angular map for the three-dimensional Lorenz system, computed with $N = 2000$, for $s = 1$ (left) and $s = 2$ (right). Each peak is labeled with its associated trace space.

For the proof we use a linear algebra lemma inspired by [7, Theorem 2.5.1].

Lemma 4.4. *Let $A \in \mathbb{R}^{d,d}$, let P be an orthogonal projector in $\mathbb{R}^d$ and let $\|\cdot\|$ be the spectral norm. Then the following equality holds*

$$(4.9) \quad \|(I - P)AP + PA^\top(I - P)\| = \|(I - P)AP\|.$$

Proof. Recall $\|B\| = \|B^\top\|$ for $B \in \mathbb{R}^{d,d}$. Further, by the orthogonality of P ,

$$\|(I - P)x + Py\|^2 = \|(I - P)x\|^2 + \|Py\|^2, \quad \forall x, y \in \mathbb{R}^d.$$

Using this and $P^\top = P$, we obtain for all $x \in \mathbb{R}^d$

$$\begin{aligned} \|((I - P)AP + PA^\top(I - P))x\|^2 &= \|(I - P)APx\|^2 + \|PA^\top(I - P)x\|^2 \\ &\leq \|(I - P)AP\|^2\|Px\|^2 + \|PA^\top(I - P)\|^2\|(I - P)x\|^2 \\ &= \|(I - P)AP\|^2(\|Px\|^2 + \|(I - P)x\|^2) = \|(I - P)AP\|^2\|x\|^2; \end{aligned}$$

hence, $\|(I - P)AP + PA^\top(I - P)\| \leq \|(I - P)AP\|$. Next, there exists some $x \in \mathbb{R}^d$ with $\|x\| = 1$ and $\|(I - P)APx\| = \|(I - P)AP\|$. Then we have $\|Px\| \leq 1$ and

$$\begin{aligned} \|(I - P)AP\| &= \|(I - P)AP(Px)\| = \|((I - P)AP + PA^\top(I - P))(Px)\| \\ &\leq \|(I - P)AP + PA^\top(I - P)\|\|Px\| \leq \|(I - P)AP + PA^\top(I - P)\|. \end{aligned}$$

Combining both estimates leads to the equality in (4.9). ■

The following proposition sharpens the result [4, Theorem 3.4] on derivatives of principal angles and simplifies the proof.

Proposition 4.5. *Let $A \in C_b^1([0, \infty), \mathbb{R}^{d,d})$. Then there exist constants $C, h_0 > 0$ such that for all $\tau \geq 0$, $V \in \mathcal{G}(s, d)$ and $0 < h \leq h_0$,*

$$(4.10) \quad \left| \frac{1}{h} \mathcal{L}(\Phi(\tau + h, \tau)V, V) - \|(I - P_V)A(\tau)P_V\| \right| \leq Ch,$$

where P_V denotes the orthogonal projector onto V , $\|\cdot\|$ is the spectral norm, and Φ is the solution operator of (4.2).

Proof. Choose an orthonormal basis of V , i.e., some $W_0 \in \mathbb{R}^{d,s}$ with $\mathcal{R}(W_0) = V$ and $W_0^\top W_0 = I_s$. For a fixed $\tau \geq 0$, define $W(t) = \Phi(\tau + t, \tau)W_0$ so that $\mathcal{R}(W(t)) = \Phi(\tau + t, \tau)V$ holds. The matrix $W(t)$ and its derivative solve the initial value problems

$$\begin{aligned} \dot{W}(t) &= A(t + \tau)W(t), \quad W(0) = W_0, \\ \ddot{W}(t) &= [\dot{A}(t + \tau) + A(t + \tau)^2]W(t), \quad \dot{W}(0) = A(\tau)W_0 =: \dot{W}_0. \end{aligned}$$

Let $\|A(\tau)\| + \|\dot{A}(\tau)\| \leq C$ for all $\tau \geq 0$. Then a Gronwall estimate yields $\|W(t)\| \leq e^{tC}\|W_0\| \leq e^C$ for $0 \leq t \leq 1$ as well as

$$\|\dot{W}(t)\| \leq Ce^C, \quad \|\ddot{W}(t)\| \leq C(1 + C)e^C =: C_\star.$$

By a Taylor expansion we obtain for $0 \leq h \leq 1$

$$\|W(h) - W_0 - h\dot{W}_0\| = \|h^2 \int_0^1 (1 - \eta)\ddot{W}(\eta h)d\eta\| \leq \frac{C_\star}{2}h^2.$$

We write this in the usual way as $W(h) = W_0 + h\dot{W}_0 + \mathcal{O}(h^2)$, where $\mathcal{O}(h^2)$ holds with a constant that is uniform in $\tau \geq 0$ and $V \in \mathcal{G}(s, d)$. Correspondingly, we have $W(h)^\top = W_0^\top + h\dot{W}_0^\top + \mathcal{O}(h^2)$. This implies

$$W(h)^\top W(h) = I_s + h(W_0^\top \dot{W}_0 + \dot{W}_0^\top W_0) + \mathcal{O}(h^2).$$

Hence, there exists a $0 < h_0 \leq 1$, independent of $\tau \geq 0$ and $V \in \mathcal{G}(s, d)$, such that the inverse exists for $0 \leq h \leq h_0$ and satisfies

$$\left(W(h)^\top W(h)\right)^{-1} = I_s - h(W_0^\top \dot{W}_0 + \dot{W}_0^\top W_0) + \mathcal{O}(h^2).$$

The orthogonal projectors associated to V and to $\Phi(\tau + h, \tau)V$, $0 \leq h \leq h_0$, are

$$\begin{aligned} P_V &= W_0 W_0^\top, \\ P_{\Phi(\tau+h, \tau)V} &= W(h) \left(W(h)^\top W(h)\right)^{-1} W(h)^\top. \end{aligned}$$

Using the previous expansions, we obtain

$$\begin{aligned} (4.11) \quad P_{\Phi(\tau+h, \tau)V} &= W_0 W_0^\top + h \left[\dot{W}_0 W_0^\top - W_0 (W_0^\top \dot{W}_0 + \dot{W}_0^\top W_0) W_0^\top + W_0 \dot{W}_0^\top \right] + \mathcal{O}(h^2) \\ &= P_V + h \left[(I - W_0 W_0^\top) \dot{W}_0 W_0^\top + W_0 \dot{W}_0^\top (I - W_0 W_0^\top) \right] + \mathcal{O}(h^2) \\ &= P_V + h \left[(I - P_V) A(\tau) P_V + P_V A(\tau)^\top (I - P_V) \right] + \mathcal{O}(h^2). \end{aligned}$$

For later reference, let us note that this implies

$$(4.12) \quad \|P_{\Phi(\tau+h, \tau)V} - P_V\| = \mathcal{O}(h).$$

Let us abbreviate $\varphi(h) = \angle(\Phi(\tau + h, \tau)V, V)$ in the following. With Lemma 4.4 and formula (2.1) we obtain from the expansion above

$$\begin{aligned} &\left| \frac{1}{h} \sin(\varphi(h)) - \|(I - P_V) A(\tau) P_V\| \right| \\ &= \left| \frac{1}{h} \|P_{\Phi(\tau+h, \tau)V} - P_V\| - \|(I - P_V) A(\tau) P_V + P_V A(\tau)^\top (I - P_V)\| \right| \\ &\leq \left\| \frac{1}{h} (P_{\Phi(\tau+h, \tau)V} - P_V) - \left[(I - P_V) A(\tau) P_V + P_V A(\tau)^\top (I - P_V) \right] \right\| \\ &= \mathcal{O}(h). \end{aligned}$$

Finally, using $\arcsin(y) = y + \mathcal{O}(y^2)$ and $\sin(\varphi(h)) = \mathcal{O}(h)$, we end up with

$$\begin{aligned} \frac{\varphi(h)}{h} &= \frac{1}{h} \arcsin(\sin(\varphi(h))) = \frac{1}{h} (\sin(\varphi(h)) + \mathcal{O}(\sin(\varphi(h))^2)) \\ &= \frac{1}{h} (\sin(\varphi(h)) + \mathcal{O}(h^2)) = \|(I - P_V) A(\tau) P_V\| + \mathcal{O}(h). \end{aligned} \quad \blacksquare$$

Proof of Theorem 4.3. Let us abbreviate $V_\tau = \Phi(\tau, 0)V$ and $E(\tau) = (I - P_{V_\tau}) A(\tau) P_{V_\tau}$.

With $T = Nh$ we obtain by substituting $\tau = (j - 1 + r)h$

$$\begin{aligned}
e(h, T, V) &= \frac{1}{hN} \left| \sum_{j=1}^N \angle(V_{(j-1)h}, V_{jh}) - \sum_{j=1}^N \int_{(j-1)h}^{jh} \|E(\tau)\| d\tau \right| \\
&= \frac{1}{N} \left| \frac{1}{h} \sum_{j=1}^N \angle(V_{(j-1)h}, V_{jh}) - \sum_{j=1}^N \int_0^1 \|E((j-1+r)h)\| dr \right| \\
&\leq \frac{1}{N} \left[\sum_{j=1}^N \left| \frac{1}{h} \angle(V_{(j-1)h}, \Phi(jh, (j-1)h)V_{(j-1)h}) - \|E((j-1)h)\| \right| \right. \\
&\quad \left. + \sum_{j=1}^N \int_0^1 \|E((j-1)h) - E((j-1+r)h)\| dr \right].
\end{aligned}$$

The terms in the first sum are bounded by $\mathcal{O}(h)$ due to (4.10) with $\tau = (j-1)h$, $V = V_{(j-1)h}$. Recall that all $\mathcal{O}$ -estimates are uniform in τ and V . The terms in the second sum are also $\mathcal{O}(h)$ due to (4.12) and the boundedness and Lipschitz boundedness of $A(\cdot)$:

$$\begin{aligned}
\|E(\tau) - E(\tau + rh)\| &\leq \|(I - P_{V_\tau}) - (I - P_{V_{\tau+rh}})\| \|A(\tau)P_{V_\tau}\| \\
&\quad + \|I - P_{V_{\tau+rh}}\| \|A(\tau) - A(\tau + rh)\| \|P_{V_\tau}\| \\
&\quad + \|I - P_{V_{\tau+rh}}\| \|A(\tau + rh)\| \|P_{V_\tau} - P_{V_{\tau+rh}}\| \\
&\leq Ch, \quad \forall \tau \geq 0, 0 \leq h \leq h_0, 0 \leq r \leq 1.
\end{aligned}$$

In view of the prefactor $\frac{1}{N}$ we finally obtain $e(h, T, V) = \mathcal{O}(h)$. ■

As in (1.5) let us introduce the angular range generated by a subspace $V \in \mathcal{G}(s, d)$ for a fixed stepsize $h > 0$:

$$(4.13) \quad I_s^{\text{step}}(h, V) = \left[\lim_{N \rightarrow \infty} \alpha_s^{\text{step}}(h, Nh, V), \overline{\lim}_{N \rightarrow \infty} \alpha_s^{\text{step}}(h, Nh, V) \right].$$

The result of Theorem 4.3 enables us to prove a strong approximation of the (outer) angular spectrum Σ_s^{cont} (see (4.5)) by its discrete counterpart

$$\Sigma_s^{\text{step}}(h) = \text{cl} \left[\bigcup_{V \in \mathcal{G}(s, d)} I_s^{\text{step}}(h, V) \right]$$

with respect to the Hausdorff metric.

Theorem 4.6. *Let $A \in C_b^1([0, \infty), \mathbb{R}^{d, d})$. Then there exist constants $C, h_0 > 0$ such that*

$$(4.14) \quad d_H(\Sigma_s^{\text{step}}(h), \Sigma_s^{\text{cont}}) \leq Ch, \quad \forall 0 < h \leq h_0,$$

where d_H denotes the Hausdorff distance.

Proof. We will show the estimate

$$d_H(I_s^{\text{step}}(h, V), I_s^{\text{cont}}(V)) \leq Ch, \quad \forall 0 < h \leq h_0, V \in \mathcal{G}(s, d)$$

for the angular ranges from (4.4) and (4.13). The estimate (4.14) then follows by taking the union over $V \in \mathcal{G}(s, d)$ and by a subsequent closure argument. As another preparation we note that $E(t) = (I - P_{\Phi(t,0)V})A(t)P_{\Phi(t,0)V}$ satisfies for $T, \tau > 0$

$$(4.15) \quad \begin{aligned} |\alpha_s^{\text{cont}}(T, V) - \alpha_s^{\text{cont}}(\tau, V)| &\leq \left| \frac{1}{T} - \frac{1}{\tau} \right| \int_0^T \|E(\zeta)\| d\zeta + \left| \frac{1}{\tau} \int_T^\tau \|E(\zeta)\| d\zeta \right| \\ &\leq \frac{2}{\tau} |T - \tau| \sup_{\zeta \geq 0} \|E(\zeta)\| \leq \frac{2C}{\tau} |T - \tau|. \end{aligned}$$

Consider first $\theta \in I_s^{\text{cont}}(V)$. By Lemma 4.2 there exists a sequence $T_n \rightarrow \infty$ with $\theta = \lim_{n \rightarrow \infty} \alpha_s^{\text{cont}}(T_n, V)$. For $0 < h \leq h_0$ choose $n_0 = n_0(h)$ such that $|\alpha_s^{\text{cont}}(T_n, V) - \theta| \leq h$ for all $n \geq n_0$. Then we set $N(h, n) = \lfloor \frac{T_n}{h} \rfloor$ so that $|T_n - hN(h, n)| \leq h$ and $N(h, n) \rightarrow \infty$ as $n \rightarrow \infty$ hold. An application of Theorem 4.3 and of (4.15) yields for $n \geq n_0$

$$\begin{aligned} |\alpha_s^{\text{step}}(h, hN(h, n), V) - \theta| &\leq |\alpha_s^{\text{step}}(h, hN(h, n), V) - \alpha_s^{\text{cont}}(hN(h, n), V)| \\ &\quad + |\alpha_s^{\text{cont}}(hN(h, n), V) - \alpha_s^{\text{cont}}(T_n, V)| + |\alpha_s^{\text{cont}}(T_n, V) - \theta| \\ &\leq Ch + \frac{2C}{T_n} |T_n - hN(h, n)| + h \leq Ch. \end{aligned}$$

Therefore, we obtain the inequalities

$$\begin{aligned} \lim_{k \rightarrow \infty} \alpha_s^{\text{step}}(h, kh, V) - Ch &\leq \lim_{n \rightarrow \infty} \alpha_s^{\text{step}}(h, hN(h, n), V) - Ch \leq \theta \\ &\leq \overline{\lim}_{n \rightarrow \infty} \alpha_s^{\text{step}}(h, hN(h, n), V) + Ch \leq \overline{\lim}_{k \rightarrow \infty} \alpha_s^{\text{step}}(h, kh, V) + Ch. \end{aligned}$$

Thus, we have shown $\text{dist}(I_s^{\text{cont}}, I_s^{\text{step}}(h)) \leq Ch$ for the Hausdorff semidistance.

The proof of the converse estimate is somewhat analogous. For $\theta \in I_s^{\text{step}}(h)$, Lemma 3.1 yields a sequence $N(h, n) \rightarrow \infty$ with $\lim_{n \rightarrow \infty} \alpha_s^{\text{step}}(h, hN(h, n), V) = \theta$. Then we choose $n_0 = n_0(h)$ with $|\alpha_s^{\text{step}}(h, hN(h, n), V) - \theta| \leq h$ for all $n \geq n_0$. Thus Theorem 4.3 leads for $n \geq n_0$ to the estimate

$$\begin{aligned} |\theta - \alpha_s^{\text{cont}}(hN(h, n), V)| &\leq |\theta - \alpha_s^{\text{step}}(h, hN(h, n), V)| \\ &\quad + |\alpha_s^{\text{step}}(h, hN(h, n), V) - \alpha_s^{\text{cont}}(hN(h, n), V)| \leq h + Ch. \end{aligned}$$

As above, we obtain

$$\begin{aligned} \lim_{T \rightarrow \infty} \alpha_s^{\text{cont}}(T, V) - Ch &\leq \lim_{n \rightarrow \infty} \alpha_s^{\text{cont}}(hN(h, n), V) - Ch \leq \theta \\ &\leq \overline{\lim}_{n \rightarrow \infty} \alpha_s^{\text{cont}}(hN(h, n), V) + Ch \leq \overline{\lim}_{T \rightarrow \infty} \alpha_s^{\text{cont}}(T, V) + Ch. \end{aligned}$$

Thus, we conclude $\text{dist}(I_s^{\text{step}}(h), I_s^{\text{cont}}) \leq Ch$, which finishes our proof. ■

Remark 4.7. Recall the dependence of the linear system (4.2) on the data $x_0 \in \mathcal{B}$. If $A(\cdot, x_0)$ and its first derivative $\dot{A}(\cdot, x_0)$ have bounds that are uniform for $t \geq 0$ and $x_0 \in \mathcal{B}$, then the estimates in Theorems 4.3 and 4.6 hold uniformly for $x_0 \in \mathcal{B}$.

4.6. A simplified algorithm for the continuous-time case. Recall the continuous-time system

$$\dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R}$$

and its solution operator $\Psi(t_1, t_0, x_0)$ from (4.1). For sufficiently small $h > 0$, we use the h -step map $\Psi(nh, (n-1)h; \cdot)$ in place of F_{n-1} of the previous algorithms. The following simple algorithm uses only points on the trajectory.

Algorithm 6 A simplified algorithm for continuous-time systems in case $s = 1$

```

1: for  $i \in \mathcal{L}^d$  do
2:   Choose the midpoint  $x_0 \in \mathcal{B}_i$ 
3:   for  $n = 1, \dots, N+1$  do
4:      $x_n = \Psi(nh, (n-1)h, x_{n-1})$ 
5:      $\Gamma_i = \frac{1}{hN} \sum_{\ell=1}^N \angle(x_{\ell-1} - x_\ell, x_\ell - x_{\ell+1})$ 
6:   Color the box  $\mathcal{B}_i$  with the color that corresponds to the angle  $\Gamma_i$ 

```

In Section 4.7 we will prove that, for autonomous systems under suitable assumptions, the resulting angular range is close to the angular range associated with the flow direction.

When applying Algorithm 6 to the ODE (4.7), we compute the differences using a continuous lift of the first coordinate to $\mathbb{R}$. Figure 4.5 shows the resulting angular values along the flow direction and the corresponding histogram in the left panels. The angular map looks similar to the upper-right diagram in Figure 4.1, where we compute the maximum angular spectral values by solving an optimization problem. The right panel of Figure 4.5 gives the pointwise absolute difference between the maximum angular spectral values and the angular values along the flow direction.

To examine these differences more closely, we consider initial points $x_0 = (\frac{\pi}{2}, x_2)^\top$, $x_2 \in [0, \pi]$, along the white dashed line in Figure 4.5. We compute the minimal and maximal angular values and compare them with the values obtained from Algorithm 6, using $T = 5000$ and $h = \frac{1}{20}$. Each step is evaluated by five Runge–Kutta steps of size $\frac{1}{100}$. Starting from a uniform grid in x_2 , we locally maximize the difference between the maximal and minimal angular values with respect to x_2 . The results in Figure 4.6 reveal three pronounced peaks in this difference. Near $x_2 = 0.236$, the flow direction yields approximately the minimal angular value, whereas near $x_2 = 2.6$ it yields approximately the maximal value. Near $x_2 = 0.733$, its angular value lies strictly between the two extrema. Thus, the mean rotation along the flow direction does not capture the full range of mean rotations of infinitesimal perturbations. The angular maps identify regions where this additional variation occurs. A detailed analysis of the associated periodic orbits will be given in a subsequent paper.

Figure 4.7 presents an application of Algorithm 6 to the three-dimensional Lorenz system (4.8) with step size $h = \frac{1}{20}$ and $N = 2000$ on a grid of size L^3 , where $L = 100$.

For the orbit segments considered here, the numerical computations in Section 4.4 yield three separated finite-time spectral intervals, with the second interval containing 1. The corresponding computed spectral fiber $\mathcal{W}_n^2$ is associated with the flow direction. Thus, in the left histogram of Figure 4.4, the peak near 8.4 corresponds to the flow direction.

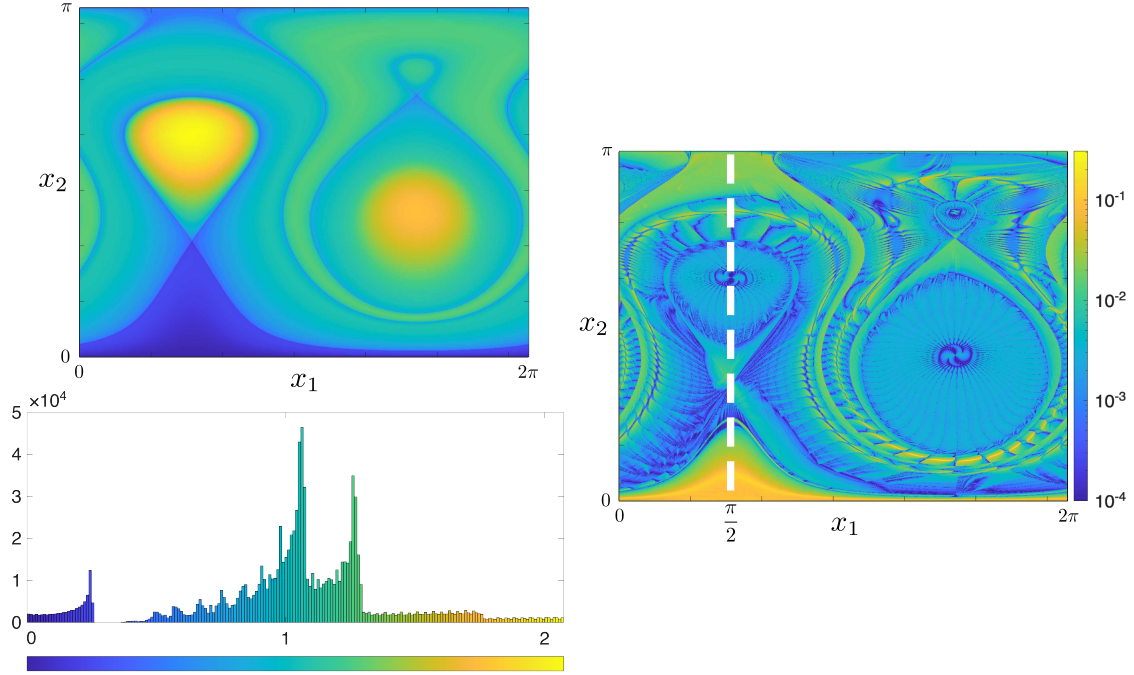


Figure 4.5: Left panels: Angular map of the flow direction for (4.7) and the corresponding histogram. The right panel shows, on a logarithmic color scale, the pointwise absolute difference between the angular map in the upper-left panel and the maximum angular spectral values shown in the upper-right panel of Figure 4.1. The white dashed line marks the section $x_1 = \frac{\pi}{2}$ investigated in Figure 4.6.

Accordingly, the simplified approach of Algorithm 6 produces a peak at approximately the same value in Figure 4.7.

4.7. Convergence analysis of Algorithm 6. In this section we consider Algorithm 6 with step-size h for an autonomous system

$$(4.16) \quad \dot{x}(t) = f(x(t)), \quad t \in \mathbb{R}, \quad x(0) = x_0.$$

Algorithm 6 aims at approximating the time average $\alpha_1^{\text{cont}}(T, V_0)$ from (4.3) for the special choice $V_0 = \text{span}(\dot{x}(0))$ by

$$(4.17) \quad \alpha_1^{\text{diff}}(h, T, V_0) = \frac{1}{T} \sum_{j=1}^N \angle(x((j-1)h) - x(jh), x(jh) - x((j+1)h)), \quad h = \frac{T}{N}.$$

Here and in what follows we suppress the dependence of the solution $x(t) = x(t; x_0)$ on the initial value. In the autonomous case, the space V_0 is mapped by the solution operator $\Phi(\tau, 0)$ of the variational equation (4.2) to

$$(4.18) \quad V_\tau = \text{span}(\dot{x}(\tau)), \quad \dot{x}(\tau) = \Phi(\tau, 0)\dot{x}(0).$$

As above, we do not discuss the error caused by the approximation of the nonlinear flow $\Psi(\tau, 0)$ in (4.17) and treat only the error induced by taking differences along the solution path. The analog of Proposition 4.5 is the following.

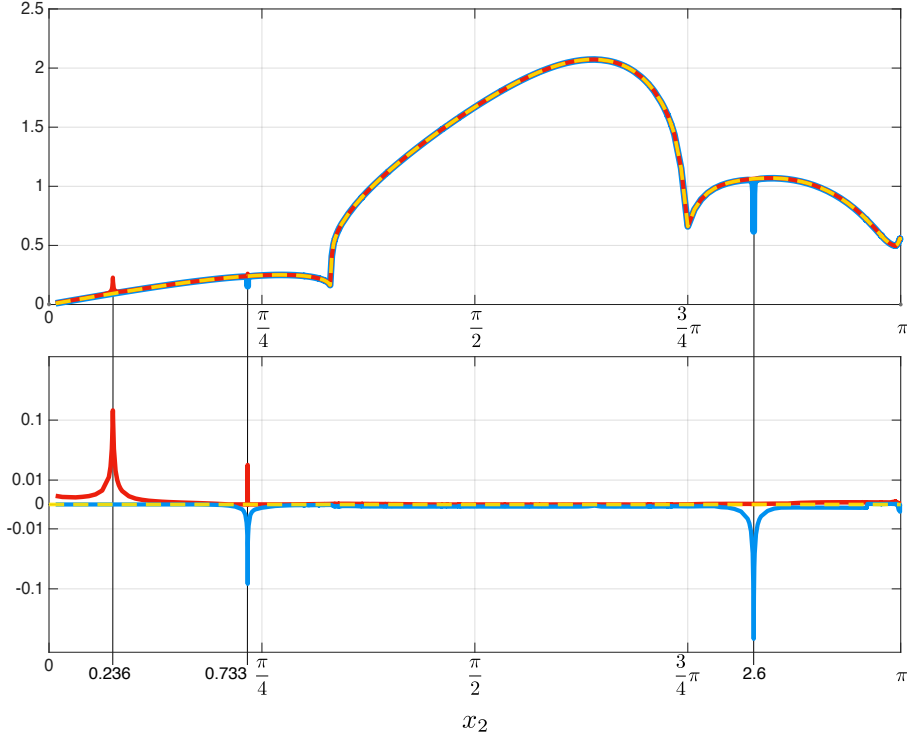


Figure 4.6: Angular values for initial points $(\frac{\pi}{2}, x_2)^\top$ along the white dashed line in Figure 4.5, with $T = 5000$ and $h = \frac{1}{20}$. Top: minimal (blue) and maximal (red) angular values, compared with the values obtained from Algorithm 6 (yellow dashed). Bottom: the same values after subtracting the angular value of the flow direction, which therefore corresponds to zero. The vertical axis uses a symmetric logarithmic scale with an approximately linear region near zero, with transition scale 10^{-2} .

Proposition 4.8. *Let $f \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ and let $x(t)$, $t \geq 0$, be a nonconstant bounded solution of (4.16). Then there exist constants $C, h_0 > 0$ such that*

$$(4.19) \quad |\angle(\dot{x}(\tau-h), \dot{x}(\tau)) - \angle(x(\tau-h) - x(\tau), x(\tau) - x(\tau+h))| \leq Ch^2$$

holds for all $0 < h \leq h_0$ and for all $\tau \geq h$.

Proof. In the following we write $P_v = P_{\text{span}(v)}$ for the orthogonal projector onto a one-dimensional subspace. From (2.1) we obtain

$$(4.20) \quad \begin{aligned} & \left| \sin(\angle(\dot{x}(\tau-h), \dot{x}(\tau))) - \sin(\angle(x(\tau-h) - x(\tau), x(\tau) - x(\tau+h))) \right| \\ &= \left| \|P_{\dot{x}(\tau-h)} - P_{\dot{x}(\tau)}\| - \|P_{x(\tau-h)-x(\tau)} - P_{x(\tau)-x(\tau+h)}\| \right| \\ &\leq \|P_{\dot{x}(\tau-h)} - P_{\dot{x}(\tau)} - P_{x(\tau-h)-x(\tau)} + P_{x(\tau)-x(\tau+h)}\|. \end{aligned}$$

Due to our assumptions we have that $\dot{x}(\tau) = f(x(\tau)) \neq 0$ is bounded for $\tau \geq 0$, and so are the derivatives $\ddot{x} = Df(x)\dot{x}$ and $\ddot{x} = D^2f(x)(\dot{x}, \dot{x}) + Df(x)\ddot{x}$. Moreover, by a Gronwall estimate on an interval of length 1 we find a constant $C > 0$ such that

$$\|\ddot{x}(t)\| + \|\ddot{x}(t)\| \leq C\|\dot{x}(\tau)\| \quad \text{for all } \tau, t \geq 0 \text{ with } |\tau - t| \leq 1.$$

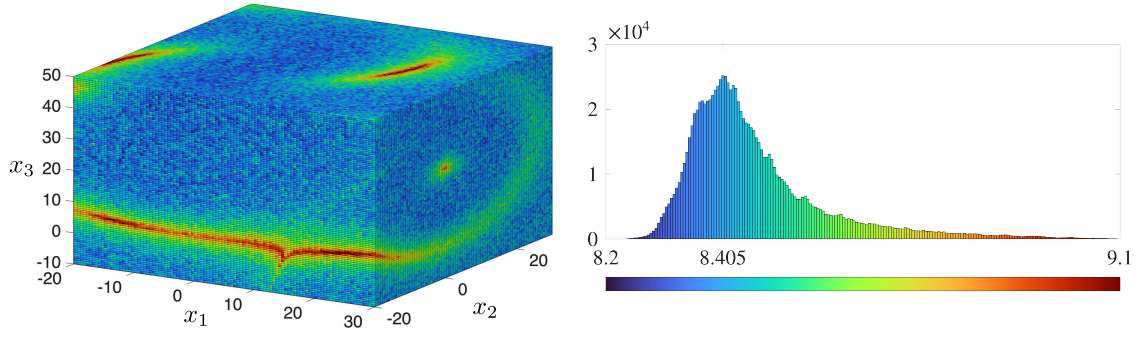


Figure 4.7: Angular map of the flow direction for (4.8) with $N = 2000$ and $h = \frac{1}{20}$ (left) and the corresponding histogram (right).

Therefore, the following Taylor expansions hold with remainders that are uniform for $0 < h \leq 1$ and $0 < h \leq \tau < \infty$:

$$\begin{aligned}
 \dot{x}(\tau - h) &= \dot{x}(\tau) - h\ddot{x}(\tau) + \mathcal{O}(h^2\|\dot{x}(\tau)\|), \\
 x(\tau + h) &= x(\tau) + h\dot{x}(\tau) + \frac{1}{2}h^2\ddot{x}(\tau) + \mathcal{O}(h^3\|\dot{x}(\tau)\|), \\
 x(\tau - h) &= x(\tau) - h\dot{x}(\tau) + \frac{1}{2}h^2\ddot{x}(\tau) + \mathcal{O}(h^3\|\dot{x}(\tau)\|).
 \end{aligned}
 \tag{4.21}$$

Note that we will not assume any lower bound for $\|\dot{x}(\tau)\|$. Using (4.21), we expand the first term on the righthand side as in (4.11):

$$\begin{aligned}
 \|\dot{x}(\tau - h)\|^2 &= \|\dot{x}(\tau)\|^2 - 2h\ddot{x}(\tau)^\top \dot{x}(\tau) + \mathcal{O}(h^2\|\dot{x}(\tau)\|^2), \\
 \|\dot{x}(\tau - h)\|^{-2} &= \|\dot{x}(\tau)\|^{-2} + 2h\ddot{x}(\tau)^\top \dot{x}(\tau)\|\dot{x}(\tau)\|^{-4} + \mathcal{O}(h^2\|\dot{x}(\tau)\|^{-2}), \\
 P_{\dot{x}(\tau-h)} &= \|\dot{x}(\tau - h)\|^{-2}\dot{x}(\tau - h)\dot{x}(\tau - h)^\top = P_{\dot{x}(\tau)} \\
 &\quad - h\|\dot{x}(\tau)\|^{-2}\left[\ddot{x}(\tau)\dot{x}(\tau)^\top + \dot{x}(\tau)\ddot{x}(\tau)^\top - 2\ddot{x}(\tau)^\top \dot{x}(\tau)P_{\dot{x}(\tau)}\right] + \mathcal{O}(h^2).
 \end{aligned}$$

In a similar fashion, we find from (4.21)

$$\begin{aligned}
 \|x(\tau \pm h) - x(\tau)\|^{-2} &= h^{-2}\|\dot{x}(\tau)\|^{-2}\left(1 \mp h\|\dot{x}(\tau)\|^{-2}\ddot{x}(\tau)^\top \dot{x}(\tau) + \mathcal{O}(h^2)\right), \\
 P_{x(\tau \pm h) - x(\tau)} &= P_{\dot{x}(\tau)} \pm h\|\dot{x}(\tau)\|^{-2}\left[\frac{1}{2}(\ddot{x}(\tau)\dot{x}(\tau)^\top + \dot{x}(\tau)\ddot{x}(\tau)^\top) \right. \\
 &\quad \left. - \ddot{x}(\tau)^\top \dot{x}(\tau)P_{\dot{x}(\tau)}\right] + \mathcal{O}(h^2).
 \end{aligned}$$

Inserting both expansions into (4.20), we obtain that the righthand side is $\mathcal{O}(h^2)$. Finally, using the Lipschitz bound for arcsin near zero, we find that the difference of the angles is $\mathcal{O}(h^2)$, as claimed in (4.19). $\blacksquare$

As a consequence of Proposition 4.8 we obtain an estimate for the approximate values (4.17) of the angular spectrum.

Theorem 4.9. *Let the assumptions of Proposition 4.8 hold. Then there exist constants $C, h_0 > 0$ such that the values $\alpha_1^{\text{diff}}(h, T, V_0)$ with initial subspace $V_0 = \text{span}(\dot{x}(0))$ satisfy*

$$|\alpha_1^{\text{cont}}(T, V_0) - \alpha_1^{\text{diff}}(h, T, V_0)| \leq Ch
 \tag{4.22}$$

for all $T > 0$ and $h = \frac{T}{N} \leq h_0$.

Proof. By Theorem 4.3 it is enough to prove the estimate (4.22) for α_1^{step} rather than α_1^{cont} . Such an estimate follows immediately from (4.18) and Proposition 4.8

$$\begin{aligned} & |\alpha_1^{\text{diff}}(h, T, V_0) - \alpha_1^{\text{step}}(h, T, V_0)| \\ &= \frac{1}{hN} \left| \sum_{j=1}^N \angle(x((j-1)h) - x(jh), x(jh) - x((j+1)h)) - \angle(\dot{x}((j-1)h), \dot{x}(jh)) \right| \\ &\leq \frac{1}{hN} \sum_{j=1}^N Ch^2 = Ch. \end{aligned}$$

This finishes the proof. ■

As in (4.13) let us introduce the angular range generated by the subspace $V_0 = \text{span}(\dot{x}(0))$ for a fixed stepsize $h > 0$:

$$I_1^{\text{diff}}(h, V_0) = \left[\liminf_{N \rightarrow \infty} \alpha_1^{\text{diff}}(h, Nh, V_0), \overline{\lim}_{N \rightarrow \infty} \alpha_1^{\text{diff}}(h, Nh, V_0) \right].$$

We conclude with the convergence of the angular ranges with respect to the Hausdorff metric. The proof uses the same arguments as in Theorem 4.6, except that we cannot proceed from the angular range to the angular spectrum since we consider only the subspace spanned by the flow direction.

Theorem 4.10. *Let the assumptions of Proposition 4.8 hold. Then there exist constants $C, h_0 > 0$ such that*

$$d_H(I_1^{\text{diff}}(h, V_0), I_1^{\text{cont}}(V_0)) \leq Ch, \quad \forall 0 < h \leq h_0$$

holds for the Hausdorff distance d_H .

Use of AI tools declaration. During the preparation of this work, the authors used ChatGPT (OpenAI) to assist with language editing, the critical examination of mathematical arguments, and the development of MATLAB code for numerical experiments. The authors reviewed and verified the resulting material and take full responsibility for the content of this paper.

Acknowledgments. Both authors are grateful to the Research Centre for Mathematical Modelling (RCM²) at Bielefeld University for continuous support of their joint research. The work of WJB was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – SFB 1283/2 2021 – 317210226. TH thanks the Faculty of Mathematics at Bielefeld University for further support.

REFERENCES

- [1] B. Aulbach and S. Siegmund. The dichotomy spectrum for noninvertible systems of linear difference equations. *J. Differ. Equations Appl.*, 7:895–913, 2001.
- [2] W.-J. Beyn, G. Froyland, and T. Hüls. Angular values of nonautonomous and random linear dynamical systems: Part I—Fundamentals. *SIAM J. Appl. Dyn. Syst.*, 21(2):1245–1286, 2022.

- [3] W.-J. Beyn and T. Hüls. Angular values of nonautonomous linear dynamical systems: Part II – Reduction theory and algorithm. *SIAM J. Appl. Dyn. Syst.*, 22(1):162–198, 2023.
- [4] W.-J. Beyn and T. Hüls. On the smoothness of principal angles between subspaces and their application to angular values of dynamical systems. *Dynamical Systems - An International Journal*, 39(3):461–499, 2024.
- [5] W.-J. Beyn and T. Hüls. Angular spectra of linear dynamical systems in discrete time. *Discrete and Continuous Dynamical Systems- Series B*, 30(6):1859–1902, 2025. arXiv2404.02694.
- [6] G. Froyland, S. Lloyd, and N. Santitissadeekorn. Coherent sets for nonautonomous dynamical systems. *Phys. D*, 239(16):1527–1541, 2010.
- [7] G. H. Golub and C. F. Van Loan. *Matrix computations*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, fourth edition, 2013.
- [8] T. Hüls. Computing stable hierarchies of fiber bundles. *Discrete Contin. Dyn. Syst. Ser. B*, 22(9):3341–3367, 2017.
- [9] V. Martins de Oliveira, D. Ciro, and I. L. Caldas. Dynamical trapping in the area-preserving Hénon map. *Eur. Phys. J. Special Topics*, 229(8):1507–1516, 2020.
- [10] J. Moser. *Stable and random motions in dynamical systems*, volume No. 77 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1973.
- [11] C. Pötzsche. *Geometric theory of discrete nonautonomous dynamical systems*, volume 2002 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010.
- [12] G. W. Stewart and J. G. Sun. *Matrix perturbation theory*. Computer Science and Scientific Computing. Academic Press, Inc., Boston, MA, 1990.